International Journal of Computational and Experimental Science and Engineering (IJCESEN)
Vol. 1-No.2 (2015) pp. 24-26
http://dergipark.ulakbim.gov.tr/ijcesen

ISSN: 2149-9144

Research Article

A New Approach for Solving Linear Fredholm Integro-Differential Equations[#]

Gül Gözde BİÇER[1], Yalçın ÖZTÜRK[2], Mustafa GÜLSU[1]

[1]Department of Mathematics, Faculty of Science, Mugla Sitki Kocman University, Mugla, Turkey
[2]Ula Ali Kocman Vocational School, Mugla Sitki Kocman University, Mugla, Turkey

* Corresponding Author : gulgozdebicer@hotmail.com

[#] Presented in "2nd International Conference on Computational and Experimental Science and Engineering (ICCESEN-2015)"

Keywords
Fredholm
integro-differantial equation
Bernoulli polynomials
Numerical solutions

Abstract: In this paper a numerical method is given for the solution of linear Fredholm integro differential equations under the mixed conditions using the Bernoulli polynomials. Finally, some experiments and their numerical solutions are given. The results reveal that this method is very effective and highly promising when compared with other numerical methods.

1 Introduction

Many physical problems are modelled by integral or integro differential equations. Historically, they have achieved great popularity among mathematicians and physicists in formulating boundary value problems of gravitation, electrostatics, fluid dynamics and scattering. It is also well known that initial-value and boundary-value problems for differential equations can often be converted into integral equations and there are usually significant advantages to be gained from making use of this conversion. Among these equations, Fredholm integro-differential equations (FIDEs) arise from various applications, like engineering, biology, medicine, economics, potential theory and many others.

The technique that we used is a numerical solution method, which is based on numerical solution of linear differential equations with variable coefficients in terms of Bernoulli polynomials.

In this study, the basic ideas of the above studies are developed and applied to the m^{th}-order linear Fredholm integro differential equation with variable coefficients

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = g(x) + \lambda_1 \int_a^b K_f(x,t)y(t)dt \quad (1)$$

for $0 < a \le x,t \le b < \infty$

under the mixed conditions,

$$\sum_{k=0}^{m-1} \left[a_{ik} y^{(k)}(a) + b_{ik} y^{(k)}(b)\right] = \mu_i \qquad (2)$$

for $i = 0,1,2,\dots,m-1$.

and the solution is expressed in the form,

$y(x) = \sum_{n=0}^{N} a_n B_n(x)$ which is a Bernoulli polynomial of degree N and a_n are unknown Bernoulli coefficients.

2 Fundamental Matrix Solution

Let us consider the m th-order Fredholm integro differential equation with variable coefficients (1) and find the matrix forms of each term of equation.

$\sum_{k=0}^{m} P_k(x) y^{(k)}(x) = g(x) + \lambda_1 \int_a^b K_f(x,t) y(t) dt$, or shortly $\sum_{k=0}^{m} P_k(x) y^{(k)}(x) = g(x) + \lambda_1 I_f(x)$ where $I_f(x) = \int_a^b K_f(x,t) y(t) dt$. Then we write the matrix form of $y(x)$ is $y(x) = \mathbf{B}(x)\mathbf{A}$

where $\mathbf{B}(x) = [B_0(x) \quad B_1(x) \quad \cdots \quad B_N(x)]$, $\mathbf{A} = [a_0 \quad a_1 \quad \cdots \quad a_N]^T$. By using the general representation of Bernoulli polynomials which is $B_N(x) = \sum_{i=0}^{N} \binom{N}{i} b_N x^{N-i}$ we can write $y(x)$ for variable t, $y(t) = \mathbf{B}(t)\mathbf{A}$ using the Maclaurin expansion,

$$K_f(x,t) = \mathbf{X}(x)\mathbf{K}_t \mathbf{X}^T(t), \quad \mathbf{K}_t = [k_{ij}^t], \tag{3}$$

$i, j = 0,1,2,\ldots, N$

and then using the Bernoulli expansion,

$$K_f(x,t) = \mathbf{B}(x)\mathbf{K}_f \mathbf{B}^T(t), \quad \mathbf{K}_f = [k_{ij}^f], \tag{4}$$

$i, j = 0,1,2,\ldots, N$

we find

$$\mathbf{X}(x)\mathbf{K}_t\mathbf{X}^T(t) = \mathbf{B}(x)\mathbf{K}_f\mathbf{B}^T(t) \Rightarrow$$
$$\mathbf{X}(x)\mathbf{K}_t\mathbf{X}^T(t) = \mathbf{X}(x)\mathbf{S}\mathbf{K}_f\mathbf{S}^\mathbf{T}\mathbf{X}^T(t) \Rightarrow$$
$$\mathbf{K}_t = \mathbf{S}\mathbf{K}_f\mathbf{S}^T \Rightarrow \mathbf{K}_f = \mathbf{S}^{-1}\mathbf{K}_t(\mathbf{S}^\mathbf{T})^{-1}.$$

On the other hand, $\mathbf{B}(x) = \mathbf{X}(x)\mathbf{S}$ using this relation we can find

$$\mathbf{X}^{(k)}(x) = \mathbf{X}(x)\mathbf{M}^k \quad \Rightarrow y^{(k)}(x) = \mathbf{X}(x)\mathbf{M}^k\mathbf{S}\mathbf{A}.$$

And then we obtain

$$I_f = \int_a^b \mathbf{B}(x)\mathbf{K}_f\mathbf{B}^T(t)\mathbf{A}dt = \mathbf{B}(x)\mathbf{K}_f\mathbf{Q}_f\mathbf{A} \tag{5}$$

where $\mathbf{Q}_f = \int_a^b \mathbf{B}^T(t)\mathbf{B}(t)dt$.

We can write $\mathbf{Q}_f$ also,

$$\mathbf{Q}_f = \int_a^b \mathbf{B}^T(t)\mathbf{B}(t)dt = \int_a^b \mathbf{S}^T\mathbf{X}^T(t)\mathbf{X}(t)\mathbf{S}dt = \mathbf{S}^T \underbrace{\left[\int_a^b \mathbf{X}^T(t)\mathbf{X}(t)dt\right]}_{\mathbf{H}_f} \mathbf{S}$$

$$\mathbf{H}_f = [h_{ij}^f(x)], \quad h_{ij}^f(x) = \int_a^b \mathbf{X}^T(t)\mathbf{X}(t)dt = \frac{b^{i+j+1} - a^{i+j+1}}{i+j+1}$$

$i, j = 0,1,2,\ldots, N$.

Then we obtain $\mathbf{Q}_f = \mathbf{S}^T\mathbf{H}_f\mathbf{S}$ using this relation we find $\mathbf{I}_f(x) = \mathbf{X}(x)\mathbf{S}\mathbf{K}_f\mathbf{Q}_f\mathbf{A}$. As a result, the matrix representation of

$$\sum_{k=0}^{m} P_k(x) y^{(k)}(x) = g(x) + \lambda_1 \int_a^b K_f(x,t) y(t) dt$$

can be given by $\sum_{k=0}^{m} \mathbf{P}_k y^{(k)} = \mathbf{G} + \lambda_1 \mathbf{I}_f$. On the other hand using the relation $y^{(k)}(x) = \mathbf{X}(x)\mathbf{M}^k\mathbf{S}\mathbf{A}$, the matrix form of the conditions given by (2) can be written as $\sum_{k=0}^{m-1} [a_{ik}\mathbf{X}(a) + b_{ik}\mathbf{X}(b)]\mathbf{M}^k\mathbf{S}\mathbf{A} = \mu_i$

$i = 0,1,2,\ldots, m-1$

3 Method Of Solution

We are ready to construct the fundamental matrix equation corresponding to Eq.(1). For this propose, firstly we write

$\sum_{k=0}^{m} P_k(x) y^{(k)}(x) = g(x) + \lambda_1 I_f(x)$ and then we can write $x = x_i = a + \frac{b-a}{N} i$, $i = 0,1,2,\ldots, N$ and then we obtain $\sum_{k=0}^{m} P_k(x_i) y^{(k)}(x_i) = g(x_i) + \lambda_1 I_f(x_i)$, $i = 0,1,2,\ldots, N$ so the fundamental matrix equation is gained $\sum_{k=0}^{m} \mathbf{P}_k y^{(k)} = \mathbf{G} + \lambda_1 \mathbf{I}_f$. Then, we obtain

$$\sum_{k=0}^{m}\mathbf{P}_k\mathbf{XM}^k\mathbf{SA}=\mathbf{G}+\lambda_1\left(\mathbf{XSK}_f\mathbf{Q}_f\mathbf{A}\right)\Rightarrow$$

$$\underbrace{\left(\sum_{k=0}^{m}\mathbf{P}_k\mathbf{XM}^k\mathbf{S}-\lambda_1\mathbf{XSK}_f\mathbf{Q}_f\right)}_{\mathbf{W}_f}\mathbf{A}=\mathbf{G} \qquad (6)$$

The fundamental matrix equation (6) for Eq.(1) corresponds to a system of (N+1) algebraic equation for the (N+1) unknown coefficients $a_0, a_1, a_2,..., a_N$. Briefly, we can write (6)

$$\mathbf{W}_f\mathbf{A}=\mathbf{G} \text{ or } \left[\mathbf{W}_f;\mathbf{G}\right] \qquad (7)$$

And briefly, the matrix form for conditions (2) is,

$$\mathbf{U}_i\mathbf{A}=[\mu_i] \text{ or } [\mathbf{U}_i;\mu_i] \quad ,i=0,1,2,...,m-1 \qquad (8)$$

To obtain the solution of Eq.(1) under the conditions (2), by replacing the rows matrices (8) by the last m rows of the matrix (7) we have the required augmented matrix or corresponding matrix equation $\mathbf{W}_f{}^*\mathbf{A}=\mathbf{G}^*$. If $rank\left(\mathbf{W}_f{}^*\right)=rank\left[\mathbf{W}_f{}^*;\mathbf{G}^*\right]=N+1$ we can write $\mathbf{A}=\left(\mathbf{W}_f{}^*\right)^{-1}\mathbf{G}^*$ Thus the coefficients a_i, $i=0,1,2,...,N$ are uniquely determined by the last equation.

4 Conclusion

Integro differential equations are usually difficult to solve analytically. In many cases, it is required to obtained the approximate solution. For this propose, the present method can be proposed. In this paper, Bernoulli polynomial approach has been used for the approximate solution of linear Fredholm integro differential equations. The proposed method is suggested as an efficient method for linear Fredholm integro differential equations

References

[1] M. Sezer, M. Gulsu, A new polynomial approach for solving difference and Fredholm integro-differential equation with mixed argument, Appl. Math. Comput. 171 (2005) 332-344.

[2] P. Natalini, A. Bernaridini, A generalization of the Bernoulli polynomials, J. Appl. Math. 3 (2003) 155–163.
[3] E. Tohidi, A.H.Bhrawy, K.Erfani, A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation, Appl. Math. Model. 37 (2013) 4283-4294.