International Journal of Computational and Experimental Science and Engineering (IJCESEN)
Vol. 2-No.1 (2016) pp. 4-8
http://dergipark.ulakbim.gov.tr/ijcesen

***ISSN:* 2149-9144**

Research Article

A comparative analysis of force-density based form-finding software to minimal surface equation#

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Presented in "2nd International Conference on Computational and Experimental Science and Engineering (ICCESEN-2015)"

Keywords
Tensile structures
Formfinding
Numerical Methods
Force-density
Minimal surface

Abstract: The shape of membrane and cable-net structures is usually modeled by geometry of minimal surfaces and its approximations. After experimenting with physical models during 1960s, computational methods were developed to find the initial geometry of tensile structures. Among the early numerical methods applied on form-finding of tensile structures was the finite difference method. An algorithm based on central finite differences combined with a nonlinear iterative process for finding the minimal surface over given stiff boundary conditions is developed and implemented in Mathematica®. The explicit 2-variable formulation as a second order quasi-linear partial differential equation with boundary condition, arising from the Euler-Lagrange area-minimizing condition, has been used for obtaining the soap film geometry. The force-density method, developed in 1970s by Linkwitz and Schek for the roof of the Olympia Stadium in Munich, found its implementation in commercial software EASY®, made by Technet GmbH, Germany. The commercial software used generates a surface which corresponds to the solution of the Laplace's equation. The form finding results obtained by these two methods are compared on some typical examples: the asymmetric hyperboloid membrane, the Concus' sine arc example and a saddle-like structure example.

1. Introduction

Membrane structures and cable nets are a very attractive choice for long-span roof structures especially in climates with low snow loads. Because of their optimum stress distribution, tensile structures use less building materials then conventional structures.
Tensile structures need to be "form-find" - their form is not an arbitrary one. During the second half of the 20th century, as engineers and architects experimented with the new type of structures, physical modeling was of great importance. The experiments were documented very detailed in [3] shown in Fig. 1 left. The shape of tensile structures corresponds to the geometry of minimal surfaces, which is a double curved surface that occupies the smallest area under certain boundary conditions - a particular form of boundary curves.

Figure 1. *Soap film experiments (left) in [3], ILEK Stuttgart (right)*

As a result of these experimental research a few experimental buildings were built, like the building of the Institute in Vaihingen – Stuttgart, Germany – a cable-net structure shown in Fig. 1 right. The experimental form finding methods reached their limits very fast – a change of scale to match the "real" structure

was almost impossible without the loss of reliability of the results. In building practice, minimal surfaces have been used because of their characteristics of a soap-film: constant stress field in all directions over the surface (isotropic stress). A soap film is from the mathematical point of view a minimal surface that can be described in variational, differential-geometric and complex analytic mathematical settings. The classical textbook on this matter is [1].

The double-curved geometry is the precondition of the membrane construction characteristics Due to its lack of stiffness, it cannot provide resistance to other loads than tension. Thanks to the special geometry of membrane structures, the membrane resists loads from every direction by increasing its pre-tension when loaded.

Wüchner and Bletzinger in [9] discuss the use of minimal surfaces as the optimal geometry in form-finding of tensile structures. They point out, that for architects, the minimal surface is favorable, because of the clear mathematical principles that these surfaces obey to. From engineering point of view, the material characteristics have to be taken in account. The membrane material is anisotropic and non-elastic, so the soap-film model has been doubted.

For these reasons, multidisciplinary research groups formulated methods that would offer a faster way to form-find tensile structures that could be used for structural analysis and implemented in commercial software. Veenendaal and Block in [4] and Lewis [10] gave an overview of the methods commonly used for form-finding of membranes and cable-nets. Most of them found their application in commercial form-finding software, e.g. the force-density method was implemented in EASY Software package that was used for obtaining the results in this work. Due to differences in their mathematical background and solution methods, the results vary. In [5], a visual comparison of the results of different form finding methods for the IL building in Stuttgart (Fig. 3) has been given. The differences are very noticeable.

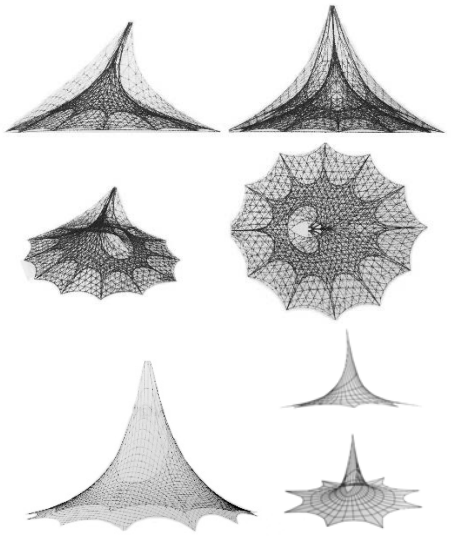

Figure 2. *Different geometries generated with several form-finding methods during the form-finding process of ILEK, Stuttgart ([5])*

2. Force-density based solution

The first solution was obtained by EASY®, software based on the Force Density Method. The method has been developed by Linkwitz and Schek in 1971 in the University of Stuttgart. A detailed description of the software package can be found in [8]. Here we give a brief explanation of the method.

The initial point for all form finding methods is the net structure of nodes and branches whose topology is described by matrix ***C,*** *branch-node matrix*. Nodes can be of two types: fixed (boundary) and free (interior). Nodes' coordinates form a coordinate vector ***x***, and the product $\boldsymbol{u} = \boldsymbol{Cx}$ represents the branches' coordinate vector, obtained as differences of coordinates of the branch ends for every rod. Branch lengths are calculated as $l = (\boldsymbol{u}^T\boldsymbol{u})^{1/2}$. Total matrices of the cable network which contain coordinates of nodes, branches and their lengths will be denoted by ***X, U*** and ***L,*** respectively. The network is in the state of equilibrium, if in each node inner forces are balanced with external loads ***p***. For a chosen node *i* in the network, let *j*, *k*, *l*, *m* be its neighbouring nodes, *a*, *b*, *c*, *d* branches determined by pairs of nodes *i-j*, *i-k*, *i-l*, *i-m* respectively. The equilibrium is expressed by:

$$\overrightarrow{f_a} + \overrightarrow{f_b} + \overrightarrow{f_c} + \overrightarrow{f_d} = \vec{p} \quad (1)$$

If we notice that

$$\overrightarrow{f_a} = f_a \cdot \frac{\overrightarrow{u_a}}{l_a} \qquad (2)$$

Where

$$\overrightarrow{u_a} = \frac{\boldsymbol{u}_a}{l_a} \qquad (3)$$

$$\overrightarrow{u_a} = (x_j - x_i)\vec{\imath} + (y_j - y_i)\vec{\jmath} + (z_j - z_i)\vec{k} \qquad (4)$$

and $\overrightarrow{u_a}/l_a$ is the unit vector of the branch *a*, we can write the equation of the equilibrium in the form

$$q_a\overrightarrow{u_a} + q_b\overrightarrow{u_b} + q_c\overrightarrow{u_c} + q_d\overrightarrow{u_d} = \vec{p} \qquad (5)$$

where the new quantities $q_a = f_a/l_a$, $q_b = f_b/l_b$, $q_c = f_c/l_c$, $q_d = f_d/l_d$ form a vector ***q.*** This quantity represents the force on the unit length of the branch, known as the force density (Schek [6]). The equilibrium of the whole cable network can be written in the following form:

$$\boldsymbol{P}_i - \boldsymbol{C}_i^{\,T}\boldsymbol{g}(u) = 0 \qquad (6)$$

$\boldsymbol{P}_i$ contains the vectors of external forces in all interior nodes, and $\boldsymbol{C}_i^{\,T}$ represents the vector of internal equivalent nodal forces for the whole network. Applying Taylor expansion to the second term, the system becomes linear, with unknown coordinates of the interior nodes

$$\boldsymbol{K}_T\Delta x = \boldsymbol{r} \qquad (7)$$

where $\boldsymbol{K}_T$ is the tangential stiffness matrix of the cable network, and ***r(u) = r(x)*** is non-balanced load in the network nodes in arbitrary iteration step.

$$\boldsymbol{K}_T = \boldsymbol{C}_i^T \left(\frac{\partial \boldsymbol{g}(\boldsymbol{u})}{\partial \boldsymbol{u}}\right)_{\boldsymbol{u}^i=\boldsymbol{u}_0} \qquad (8)$$

Vector $\boldsymbol{u} = \boldsymbol{u}_0$ contains branch coordinates of the preceding equilibrium state of the network. System in Eq. 8 leads to an iterative process:

$$\boldsymbol{K}_T^{(i)} \Delta \boldsymbol{x}^{(i+1)} = \boldsymbol{r}^{(i)} \qquad (9)$$

The iteration is continued until the residual load is zero $\boldsymbol{r}^{(i)} = 0$ up to a given precision, which leads to equilibrium state in all nodes of the cable network. To solve the system a standard or modified Newton-Raphson iterative method has being used.

3. Non-linear iterative solution of the minimal surface equation

In this analysis, the explicit 2-variable formulation as a second order quasi-linear partial differential equation with boundary condition, arising from the Euler-Lagrange area-minimizing condition, has been used. For a given domain $U \subset \mathbb{R}^2$ with a compact boundary $B = \partial U$ we are looking for a function $z = f(x,y): U \to \mathbb{R}$ which satisfies the equation

$$(1 + f_x^2) f_{yy} + \left(1 + f_y^2\right) f_{xx} - 2 f_x f_y f_{xy} = 0 \qquad (10)$$

As a model example, we take a rectangular domain $U = (a,b) \times (c,d)$ bounded by a rectangle $B = \{a\} \times (c,d) \cup \{b\} \times (c,d) \cup (a,b) \times \{c\} \cup (a,b) \times \{d\}$. An equidistant rectangular mesh with $n+1$ points along x-axis and $m+1$ points along y-axis, with steps h and k respectively, division points $x_i = a + ih, y_j = c + jk$ and mesh knots $P_{ij} = P_{ij}(x_i, y_j)$ $(i = 0, \dots, n,\ j = 0, \dots, m)$ has been layered over the domain. For approximation purposes of partial derivatives in the inner points central finite differences have been used:

$$\begin{aligned}
f_x(P_{ij}) &\approx \frac{1}{2h}\left(f_{i+1,j} - f_{i-1,j}\right) \\
f_y(P_{ij}) &\approx \frac{1}{2k}\left(f_{i,j+1} - f_{i,j-1}\right) \\
f_{xx}(P_{ij}) &\approx \frac{1}{h^2}\left(f_{i+1,j} - 2f_{i,j} + f_{i-1,j}\right) \\
f_{yy}(P_{ij}) &\approx \frac{1}{k^2}\left(f_{i,j+1} - 2f_{i,j} + f_{i,j-1}\right) \\
f_{xy}(P_{ij}) &\approx \frac{1}{4hk}\begin{pmatrix} f_{i+1,j+1} - f_{i-1,j+1} - \\ f_{i+1,j-1} + f_{i-1,j-1} \end{pmatrix}
\end{aligned} \qquad (11)$$

The subscripts represent the values of the function f in mesh points $f_{i,j} = f(P_{ij}) = f(x_i, y_j)$. The approximation error has second order with respect to h, k. This gives rise to a system of (not linear, but cubic) equations with $(n-1) \times (m-1)$ indeterminate values of f in the interior mesh points. Due to original approximation, this nonlinear system transforms into an iteration process

$$f_{i,j} = \frac{1}{2\left(4h^2 + \left(f_{i+1,j} - f_{i-1,j}\right)^2\right) + 2\left(4k^2 + \left(f_{i,j+1} - f_{i,j-1}\right)^2\right)} \times \left[\left(4h^2 + \left(f_{i+1,j} - f_{i-1,j}\right)^2\right)\left(f_{i,j+1} + f_{i,j-1}\right) + \left(4k^2 + \left(f_{i,j+1} - f_{i,j-1}\right)^2\right)\left(f_{i+1,j} + f_{i-1,j}\right) - -\frac{1}{2}\left(f_{i+1,j} - f_{i-1,j}\right)\left(f_{i,j+1} - f_{i,j-1}\right)\left(f_{i+1,j+1} - f_{i-1,j+1} - f_{i+1,j-1} + f_{i-1,j-1}\right)\right] \qquad (12)$$

To find the value in one mesh point, 8 neighboring mesh points are being used (horizontal, vertical and diagonal points). Although this is in the spirit of the method used in [2], this method is a non-linear, non-Newton iteration method.

As the initial value $f^{(0)}$ for f in the iterative process the solution of the Laplace's equation with the same boundary condition has been used. Using the central difference formulas, this equation converts to a simple linear system in the standard iterative form:

$$f_{xx} + f_{yy} = 0 \rightarrow$$
$$f_{ij} = \frac{1}{4}\left(f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1}\right) \qquad (13)$$

The boundary conditions (BC) used in calculated examples are linear. The value $f(x,y)$ is interpreted as the height of the spatial point over $(x,y) \in B$.

4. Results

The first example is an asymmetric surface over $U = (-1,1) \times (-1,1)$. High points of the membrane modelled have different heights: $z = 8$ and $z = 16$, divided into rectangular mesh with $n = m = 21$. The process described in 3 was programmed in Mathematica® and it converges to a solution of the minimal surface equation. The initial approximation $f^{(0)}$ is a hyperbolic-paraboloid -the solution of the Laplace's equation obtained in 761 iterations with precision of $\varepsilon = 10^{-5}$.

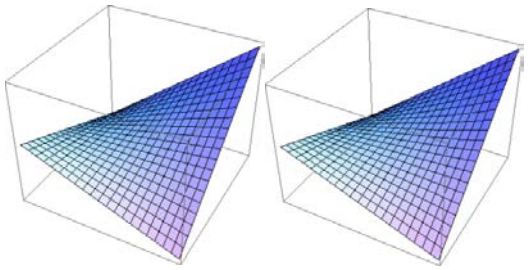

Figure 3. *The initial approximation $f^{(0)}$ (left) and the minimal surface f (right)*

After obtaining the initial shape approximation $f^{(0)}$, the minimal surface f generated is shown in Fig. 5, obtained in 427 iterations with iteration precision of $\varepsilon = 10^{-5}$. The maximum of the difference between $f^{(0)}$ and $f$over the given domain is $\max_U |f - f^{(0)}| = 0.2390$ (Fig. 6 left). Using the commercial software EASY®, in the same example (same boundary conditions without

external loads), the surface g is obtained. This surface is practically the same as $f^{(0)}$ (see Fig. 5 left) obtained by solution of the Laplace's equation. The difference of the two over the given domain is $\max_U |g - f^{(0)}| = 0.00789$.

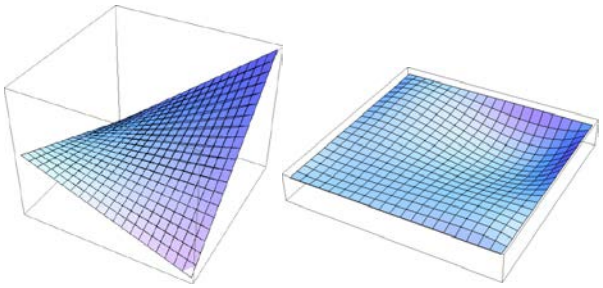

Figure 4. *The minimal surface f and surface g obtained by FD-method (left) and their difference $f - g$ (right)*

Fig. 6 left shows both the minimal surface f and the surface g found by EASY® - based on Force-density on the same graph. As it is seen, there are differences between these solutions, which can be noted especially in the region of the higher curvature - close to the higher corner. Fig. 6 right shows the difference between the minimal surface f found by the method described in 3 and the surface g. The maximum difference of the two solutions over the given domain is $\max_U |f - g| = 0.23942$. This is approximately 1.50% of the height.

The second example is a model example from [2] - all boundary lines are equal zero, except one - a sinusoidal arc. The approximation $f^{(0)}$ is obtained in 670 iterations with iteration precision of $\varepsilon = 10^{-5}$, and the minimal surface f in 620 iterations with the same precision. The maximum of the difference between $f^{(0)}$ and f over the given domain is $\max_U |f - f^{(0)}| = 0.1990$, which is shown in Fig. 7, the difference between them multiplied by factor 10.0.

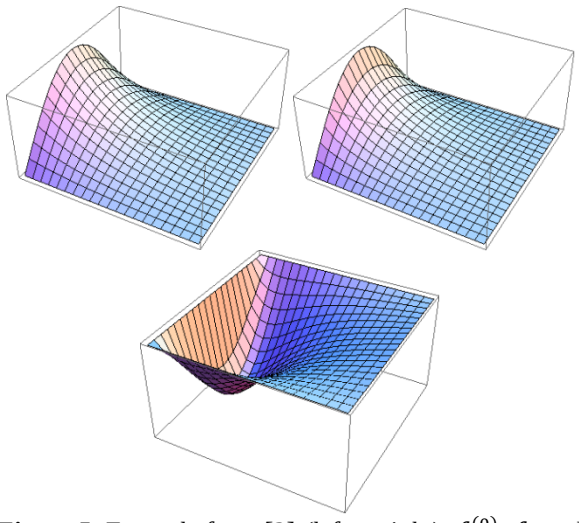

Figure 5. *Example from [2] (left to right): $f^{(0)}$, f, and $f - f^{(0)}$.*

In the third example we examined a saddle-like structure. The initial approximation $f^{(0)}$ (the solution of the Laplace's equation) is obtained in 726 iterations with iteration precision of $\varepsilon = 10^{-5}$, and the minimal surface f in 761 iterations with the same precision. The maximum of the difference between $f^{(0)}$ and f over the given domain is $\max_U |f - f^{(0)}| = 0.2011$. (shown in Fig. 8).

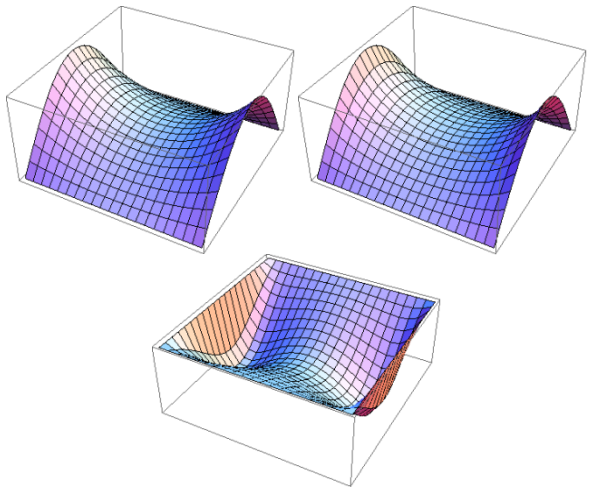

Figure 5. *A saddle-like structure with sinusoidal boundaries (left to right): $f^{(0)}$, f and $f - f^{(0)}$.*

5. Conclusion

The standard model of the soap film surface developed in architecture by Frei Otto led to a widely spread belief that the best architectural model for a membrane structure is the minimal surface (soap-film). This turned out to be controversial and contemporary engineers and researchers discuss, trying to find the optimal mathematical model for form-finding of membrane structures.

In the present paper, a simple iterative nonlinear process was used to obtain the exact minimal surface over a fixed given quadrilateral domain. The commercial software used for the same purpose as its output usually does not give the minimal surface but the surface which corresponds to the solution of the Laplace's equation. Three analyzed examples are shown: the asymmetric hyperboloid, the Concus' sine arc example and a saddle-like structure example. The difference of the two approaches did not exceed 1.50% of the height in three analyzed examples. Under standard exploiting conditions, this is not a notable difference. Under heavy load circumstances, the difference might be considered notable. It would be of interest to examine the difference of the surfaces obtained under external load.

Acknowledgement

The author expresses her sincere gratitude to Jürgen Holl and Bernd Stary from Technet GmbH for time-limited free use of the software package EASY® for research purposes and she acknowledges the support of the Ministry of Education, Science and Technical Development of the Republic of Serbia under grant OI 174020.

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