International Journal of Computational and Experimental Science and ENgineering (IJCESEN)

Vol. 11-No.4 (2025) pp. 9836-9841
http://www.ijcesen.com

ISSN: 2149-9144

Research Article

Multipliers in Almost (m, m+1)-semilattice

E. A. Geoferlin Jesho1[1]*, A Pramila Inpa Rose[2]

[1]Research scholar (Reg no: 21113112091010), Department of Mathematics, Nesamony Memorial Christian College (Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli -627 012), Marthandam-629 165, Tamilnadu, India.
* **Corresponding Author Email:** geoferlinjeshh@gmail.com - **ORCID:** 0009-0004-7333-5458

[2]Associate Professor and Head Department of Mathematics, Nesamony Memorial Christian College (Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli -627 012), Marthandam-629 165, Tamilnadu, India.
Email: pramilapream@gmail.com - **ORCID:** 0000-0002-5247-8850

Article Info:

DOI: 10.22399/ijcesen.4499
Received : 15 October 2025
Revised : 15 December 2025
Accepted : 20 December 2025

Keywords

Semilattice,
Almost (m,n)-semilattice,
Almost (m,n)-semilattice with zero,
Almost semilattice,
Multipliers,
Order relation.

Abstract:

The concept of multipliers in an Almost (m, m+1)-semilattice is introduced and we prove some important results of Almost (m, m+1)-semilattice. In this paper, we introduced the definition for an order relation and almost (m,n)-semilattice with zero. Also we proved some important remarks, lemmas and properties are discussed with examples.

1. Introduction

The concept of semilattice was introduced by F.Klein in1939.The concept of Almost Semilattice (ASL) was introduced be G.Nanaj Rao and T.G.Beyene [8] as a generalization of almost distributive lattice as a semilattice. Velrajan.M and Pramila Inpa Rose.A [10] introduced the concept of (m, n)-semilattice as a generalization of semilattice. A commutative semigroup S is called an (m, n)-semilattice if $x^n = x^m$, for all x ∈ S, where m and n are positive integers with m < n. For the background of semilattices, we refer the reader to [3, 5, 6]. The concept of Almost (m, n)-semilattice was introduced by E.A. Geoferlin Jesho and Dr. A. Pramila Inpa Rose as a generalization of semilattices [4]. Beyene, T. G., & Alemu, H. T. introduced the concept of multipliers in Almost semilattice[9].

In this paper, we introduce the concept of multipliers in an Almost (m, m+1)-semilattice which is a generalization of semilattice and derive some properties of Almost (m, m+1)-semilattice. Moreover, we prove that definition for an order relation and almost (m,n)-semilattice with zero with examples

2. Preliminaries

Definition 2.1. A partially ordered set or simply a poset (P, ≤) is called a **lattice** if l.u.b.{x,y} and g.l.b.{x,y} exist for all x, y ∈ P.

Definition 2.2. A **semilattice** S is a nonempty set and ∘ is a binary operation on S satisfying:

1. x∘ (y∘z) = (x∘y) ∘z (Associative law)
2. x∘y = y∘x (Commutative law)
3. x∘x = x, for all x , y, z ∈ S. (Idempotent law)

Definition 2.3. [8] An **Almost Semilattice** (L, ∘) is a non-empty set and ∘ is a binary operation on L, satisfies the following conditions:

(1) (x∘y)∘z = x∘(y∘z) (Associative law)

(2) $(x \circ y) \circ z = (y \circ x) \circ z$ (Almost Commutative law)

(3) $x \circ x = x$, for all $x, y, z \in L$. (Idempotent law).

Definition 2.4. [10] Let m and n be positive integers with m < n. A non empty set S together with a commutative and associative binary operation is called an **(m, n)-semilattice** if for each $a \in S$, $a^n = a^m$, where $a^n = aaa...a$ (n times).

Definition 2.5. [4] Let m and n be positive integers with m < n. A non empty set N together with a binary operation ∘ is said to be an **Almost (m, n)-semilattice** if the following axioms holds:

(i) $(x \circ y) \circ z = (y \circ x) \circ z$ (Almost commutative law)
(ii) $x \circ (y \circ z) = (x \circ y) \circ z$. (Associative law)
(iii) $x^n = x^m$,
(i.e) $x^n = x \circ x \circ x ... \circ x$ (n times), ((m,n)-potent law)
for $x, y, z \in N$.

3. Main results

3.1.Order relation on Almost (m, n)-semilattice

In this section, we define the order relation on Almost (m, n)-semilattice and study some important properties. Also, we prove theorems and related corollaries on it.

Definition 3.1.1 Let N be an Almost (m, n)-semilattice. Then for any $a, b \in N$, we say that a is less than or equal to b and write $a \leq b$, if $\mathbf{a = a \circ b}$.

Property 3.1.2 Let N be an Almost (m, n)-semilattice. If $a \leq b$, then $a \circ b = b \circ a$ for any $a, b \in N$.

Proof: Now, $a \leq b \Rightarrow a = a \circ b$

Consider, $a \circ b = (a \circ b) \circ b$
$= (b \circ a) \circ b$ [By Almost commutative law]
$= b \circ (a \circ b)$ [By Associative law]
$= b \circ a$

Theorem 3.1.3 Let N be an Almost (m, n)-semilattice. Then,

1) $\leq$ is antisymmetric and transitive .
2) Reflexive if and only if a is idempotent.

Proof:

1) **Antisymmetric**: Let $a, b \in N$.

If $a \leq b \Rightarrow a = a \circ b$ and $b \leq a \Rightarrow b = b \circ a$.

By property 3.1.2, $a \circ b = b \circ a$

Thus, a = b.

Transitive: Let $a, b, c \in N$.

Now, $a \leq b \Rightarrow a = a \circ b$ and $b \leq c \Rightarrow b = b \circ c$.

Now, $a = a \circ b$
$= a \circ (b \circ c)$
$= (a \circ b) \circ c$ [By Associative law]
$= a \circ c$.

Therefore, $a \leq c$.

2) Now, $a \leq a \Leftrightarrow a = a \circ a$
$\Leftrightarrow a = a^2$
$\Leftrightarrow$ a is idempotent.

Theorem 3.1.4 Let N be an Almost (m, n)-semilattice. Then, for any $a, b \in N$, we have the following :

(1) If $a \leq b$, then $b \circ a \leq b$.
(2) If $a \leq b$, then $(a \circ b) \circ b = b \circ a$.
(3) If $a \leq b$, then $a \circ b \leq b$.

Proof: Now, $a \leq b \Rightarrow a = a \circ b$

(1) Consider, $(b \circ a) \circ b = b \circ (a \circ b)$ [By Associative law]
$= b \circ a$

Therefore, $b \circ a \leq b$.

(2) Consider, $(a \circ b) \circ b = (b \circ a) \circ b$ [By Almost commutative law]
$= b \circ (a \circ b)$ [By Associative law]
$= b \circ a$

Therefore, $(a \circ b) \circ b = b \circ a$.

(3) From (1), $b \circ a \leq b$.
$\Rightarrow a \circ b \leq b$, by property 3.1.2.

Remark 3.1.5 From the above theorem, (1) and (2) are equivalent, and (1) and (3) are equivalent.

Proof:

Assume (1) .
(i.e) If $a \leq b$, then $b \circ a \leq b$
$\Rightarrow b \circ a = (b \circ a) \circ b$.
Consider, $(a \circ b) \circ b = (b \circ a) \circ b$
$= b \circ a$.
Hence, (2).

Now, assume (2).
i.e) If $a \leq b$, then $(a \circ b) \circ b = b \circ a$.
Consider, $(b \circ a) \circ b = (a \circ b) \circ b$
$= b \circ a$.
Hence, (1).

Therefore, (1)⇔(2).

Also, assume (1).
(i.e) If $a \leq b$, then $b \circ a \leq b$

⇒ b∘a = (b∘a)∘b.
Consider, (a∘b)∘b = (b∘a)∘b
= b∘a
= a∘b . [By property 3.1.2]
Hence, (3).
Now, assume (3).
(i.e) If a ≤ b, then a∘b ≤ b
⇒ a∘b = (a∘b)∘b.
Consider, (b∘a)∘b = (a∘b)∘b
= a∘b
= b∘a . [By property 3.1.2]
Hence, (1).
Therefore, (1)⇔(3).

Theorem 3.1.6 Let N be an Almost (m, n)-semilattice. Then, for any a, b, c, d ∈ N, we have the following:

i. If a ≤ b and c ≤ d, then a∘c ≤ b∘d.
ii. If c ≤ a and c ≤ b, then c ≤ a∘b.
iii. If c ≤ b, then a∘c ≤ b.
iv. If a ≤ b and a ≤ c, then a∘c ≤ b∘c.
v. If a ≤ b and a ≤ c, then c∘a ≤ c∘b.

Proof: Let a, b, c, d ∈ N.

i. Now, a ≤ b ⇒ a= a∘b and c ≤ d ⇒ c= c∘d.
Consider, (a∘c) ∘(b∘d) = a∘[c∘(b∘d)] [By Associative law]
= a∘[(c∘b)∘d] [By Associative law]
= a∘[(b∘c)∘d] [By Almost commutative law]
= a∘[b∘(c∘d)] [By Associative law]
= (a∘b)∘(c∘d) [By Associative law]
= a∘c .
Therefore, a∘c ≤ b∘d
ii. Now, c ≤ a ⇒ c= c∘a and c ≤ b ⇒ c= c∘b.
Consider, c ∘(a∘b) = (c∘a)∘b [By Associative law]
= c∘b
= c.
Therefore, c ≤ a∘b
iii. Now, c ≤ b ⇒ c= c∘b.
Consider, (a∘c)∘b = a∘(c∘b) [By Associative law]
= a∘c .
Therefore, a∘c ≤ b.
iv. Now, a ≤ b ⇒ a= a∘b and a ≤ c⇒ a= a∘c.
Consider, (a∘c)∘(b∘c) = a∘[c∘(b∘c)] [By Associative law]
= a∘[(c∘b)∘c] [By Associative law]
= a∘[(b∘c)∘c] [By Almost commutative law]
= [a∘(b∘c)]∘c [By Associative law]
= [(a∘b)∘c]∘c [By Associative law]
= (a∘c)∘c
= a ∘c
Therefore, a ∘c ≤ b∘c.
v. Now, a ≤ b ⇒ a= a∘b and a ≤ c⇒ a= a∘c.
Consider, (c∘a)∘(c∘b) = c∘[a∘(c∘b)] [By Associative law]
= c∘[(a∘c)∘b] [By Associative law]
= c∘(a∘b)
= c∘a
Therefore, c∘a ≤ c∘b.

Property 3.1.7 Let N be an Almost (m, n)-semilattice. If a ≤ b, then a∘a ≤ b for any a, b ∈ N.
Proof: Let a, b ∈ N.
Now, a ≤ b ⇒ a = a∘b
Consider, (a∘a)∘b = a∘(a∘b) [By Associative law]
= a∘a.
Therefore, a∘a ≤ b.

3.2. Almost (m, n)-semilattice with zero

In this section, we introduce the concept of an Almost (m, n)-semilattice with zero. And, we prove some properties and theorems with examples on Almost (m, n)-semilattice with zero.

Definition 3.2.1 A non empty set N together with a binary operation ∘ is said to be an **Almost (m, n)-semilattice with zero** if the following axioms hold:

i. (x∘y)∘z = (y∘x)∘z (Almost commutative law)
ii. x∘(y∘z) = (x∘y)∘z. (Associative law)
iii. $x^n = x^m$,
(i.e) x^n = x ∘ x ∘ x...∘ x (n times). ((m, n)-potent law)
iv. 0∘x = 0,
for x, y, z ∈ N.

Example 3.2.2 Let N = {0, a, b, c}. Define a binary operation ∘ on N as follows:

∘	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	c
c	0	a	b	a

Here, N is an Almost (2,3)-semilattice with 0.

Lemma 3.2.3 Let N be an Almost (m, n)-semilattice with 0. Then for any $a \in N$, $a \circ 0 = 0$.
Proof: Let $a \in N$.
Suppose N be an Almost (m, n)-semilattice with 0.

Then, $a \circ 0 = a \circ (0 \circ a)$ [By definition 3.2.1]
$= (a \circ 0) \circ a$ [By Associative law]
$= (0 \circ a) \circ a$ [By Almost commutative law]
$= 0 \circ a$
$= 0$
Therefore, $a \circ 0 = 0$

Lemma 3.2.4 Let N be an Almost (m, n)-semilattice with 0. Then for any $a, b \in N$, if $a \leq b$ then $a \circ b = 0$ if and only if $b \circ a = 0$.
Proof: Let $a \in N$ and $a \leq b \Rightarrow a = a \circ b$.
Suppose $a \circ b = 0$.
Then, $b \circ a = b \circ (a \circ b)$ [By definition 3.2.1]
$= (b \circ a) \circ b$ [By Associative law]
$= (a \circ b) \circ b$ [By Almost commutative law]
$= 0 \circ b$
$= 0$
Therefore, $b \circ a = 0$.
Conversely, assume $b \circ a = 0$
Now, $a \circ b = (a \circ b) \circ b$
$= (b \circ a) \circ b$ [By Almost commutative law]
$= 0 \circ b$
$= 0$
Therefore, $a \circ b = 0$.

Corollary 3.2.5 Let N be an Almost (m, n)-semilattice with 0. Then for any $a, b \in N$, if $a \leq b$ then $a \circ b = b \circ a$.
Proof: Let $a, b \in N$ and $a \leq b$.
If $a \circ b = 0$,
by lemma 3.2.4, $b \circ a = 0$.
Therefore, $a \circ b = b \circ a$.

3.3. Multipliers in Almost (m, m+1)-semilattice:

In this section, we introduce the concept of multipliers in an Almost (m, m+1)-semilattice and give examples for it. Also, we prove some related properties of multipliers in Almost (m, m+1)-semilattice. Further, we define some definitions based on an Almost (m, m+1)-semilattice and also prove some important properties based on it.

Definition 3.3.1 Let N be an Almost (m, m+1)-semilattice. A function $f : N \to N$ is called a **multiplier** of N if $f(x \circ y) = x \circ f(y)$ for all $x, y \in N$.

Example 3.3.2 Let N = {a, b, c, d}. Define a binary operation ∘ on N as follows:

∘	a	b	c	d
a	a	a	a	a
b	a	b	c	b
c	a	c	a	c
d	a	d	c	d

Then, clearly (N, ∘) is an Almost (2,3)-semilattice.

Now, define $f : N \to N$ by

$$f(y) = \begin{cases} a, & \text{when } y = a, c \\ b, & \text{when } y = b, d \end{cases}$$

Then, f is a multiplier on N.

Lemma 3.3.3 Let N be an Almost (m, m+1)-semilattice. Then the identity map on N is a multiplier on N. This multiplier is called the **identity multiplier** on N.
Proof: Let $f : N \to N$ defined by $f(y) = y$, for all $y \in N$ be the identity map on N.
Then, for any $x, y \in N$.
Now, $f(x \circ y) = x \circ y$
$= x \circ f(y)$.
Therefore, f is a multiplier on N.

Example 3.3.4 Let N= {a, b, c, d}. Define a binary operation ∘ on N as follows:

∘	a	b	c	d
a	a	a	a	a
b	a	c	a	c
c	a	a	a	c
d	a	c	c	d

Then, clearly (N, ∘) is an Almost (3,4)-semilattice.
Now, define $f : N \to N$ by $f(y) = y$, when y = a, b, c, d.
Then, f is an identity multiplier on N.

Example 3.3.5 Let N be an Almost (m, m+1)-semilattice with 0.Then a function $f : N \to N$ defined by $f(y) = 0$ for any $y \in N$ is a multiplier on N which is called a **zero multiplier**.

Remark 3.3.6 In an Almost (m, m+1)-semilattice N, every function may not be a multiplier on N.

Example 3.3.7 Let N = {u, v, w}. Define a binary operation ∘ on N as follows:

∘	u	v	w
u	u	u	u
v	u	v	w
w	u	v	w

Then, clearly (N, ∘) is an Almost (1,2)-semilattice.
Now, define $f : N \to N$ by $f(u) = w$, $f(v) = v$ and $f(w) = u$.
Then, f is not a multiplier on N, since $f(u \circ v) = f(u) = w \neq u = u \circ f(v)$.

Lemma 3.3.8 Let N be an Almost (m, m+1)-semilattice and $a \in N$. Define a function $f_a : N \to N$ by $f_a(y) = a \circ y$ for all $y \in N$. Then, f_a is a multiplier on N. Such a multiplier of N is called a **principal multiplier** of N.

Proof: Let $x, y \in N$.
Then, $f_a(x\circ y) = a\circ(x\circ y)$
$= (a\circ x)\circ y$ [By Associative law]
$= (x\circ a)\circ y$ [By Almost commutative law]
$= x\circ(a\circ y)$ [By Associative law]
$= x\circ f_a(y)$.
Therefore, f_a is a multiplier on N

Definition 3.3.9 A multiplier f in an Almost (m, m+1)-semilattice N is called an **isotone multiplier** if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in N$.

Lemma 3.3.10 Let N be an Almost (m, m+1)-semilattice and f be a multiplier of N. If $x \leq y$ and $f(y) = y$, then $f(x) = x$.

Proof: Let $x, y \in N$ such that $x \leq y$, then $x=x\circ y$.
Then, $f(x) = f(x\circ y)$
$= x\circ f(y)$
$= x\circ y$
$= x$
Therefore, $f(x) = x$.

Theorem 3.3.11 Let N be an Almost (m, m+1)-semilattice and f be a multiplier of N. If $x \leq y$, then $f(x)\circ f(y) = f(y)\circ f(x)$.

Proof: Let $x, y \in N$ such that $x \leq y$, then $x= x\circ y$.
Now, $f(x)\circ f(y) = f(x\circ y)\circ f(y)$
$= [x\circ f(y)]\circ f(y)$
$= [f(y)\circ x]\circ f(y)$ [By Almost commutative law]
$= f(y)\circ[x\circ f(y)]$ [By Associative law]
$= f(y)\circ f(x\circ y)$
$= f(y)\circ f(x)$
Therefore, $f(x)\circ f(y) = f(y)\circ f(x)$.

Lemma 3.3.12 Let N be an Almost (m, m+1)-semilattice and f_a is a principal multiplier on N. If $x \leq y$, then $f_a(x)\circ f_a(y) = f_a(y)\circ f_a(x)$.

Proof: Let $x, y \in N$ such that $x \leq y$, then $x = x\circ y$.
Now, $f_a(x)\circ f_a(y) = (a\circ x)\circ(a\circ y)$
$= (x\circ a)\circ(a\circ y)$ [By Almost commutative law]
$= [(x\circ y)\circ a]\circ(a\circ y)$ [$\because x = x\circ y$]
$= [(y\circ x)\circ a]\circ(a\circ y)$ [By Almost commutative law]
$= [y\circ(x\circ a)]\circ(a\circ y)$ [By Associative law]
$= \{y\circ[(x\circ a)\circ a]\}\circ y$ [By Associative law]
$= \{y\circ[(a\circ x)\circ a]\}\circ y$ [By Almost commutative law]
$= \{[y\circ(a\circ x)]\circ a\}\circ y$ [By Associative law]
$= \{a\circ[y\circ(a\circ x)]\}\circ y$ [By Almost commutative law]
$= \{(a\circ y)\circ(a\circ x)\}\circ y$ [By Associative law]
$= (a\circ y)\circ [(a\circ x)\circ y]$ [By Associative law]
$= (a\circ y)\circ [a\circ(x\circ y)]$ [By Associative law]
$= (a\circ y)\circ(a\circ x)$ [Since $x = x\circ y$]
$= f_a(y)\circ f_a(x)$
Therefore, $f_a(x)\circ f_a(y) = f_a(y)\circ f_a(x)$.

Lemma 3.3.13 Let N be an Almost (m, m+1)-semilattice and f_a is a principal multiplier on N. If $x \leq y$ and $f_a(y) = y$, then f_a is an isotone multiplier on N.

Proof: Let $x, y \in N$.
Now, $f_a(x)\circ f_a(y) = (a\circ x)\circ y$
$= a\circ(x\circ y)$
$= a\circ x$
$= f_a(x)$.
$\Rightarrow f_a(x) \leq f_a(y)$.
Therefore, f_a is an isotone multiplier on N.

Proposition 3.3.14 Let N be an Almost (m, m+1)-semilattice and f_1 and f_2 be the two multipliers of N. Then, the composition map of f_1 and f_2, $f_1\circ f_2$ is a multiplier of N.

Proof: Let $x, y \in N$.
Then, $(f_1\circ f_2)(x\circ y) = f_1(f_2(x\circ y))$
$= f_1(x\circ f_2(y))$ [Since f_2 is a multiplier]
$= x\circ f_1(f_2(y))$ [Since f_1 is a multiplier]
$= x\circ(f_1\circ f_2)(y)$
Therefore, $f_1\circ f_2$ is a multiplier of N.

Proposition 3.3.15 Let N_1 and N_2 be two Almost (m, m+1)-semilattices with 0. Define $f : N_1\times N_2 \rightarrow N_1\times N_2$ by $f(x, y)= (x, 0)$ for all $(x, y) \in N_1\times N_2$. Then f is a multiplier of $N_1\times N_2$ with respect to pointwise operation.

Proof: Let N_1 and N_2 be two Almost (m, m+1)-semilattices with 0.
Then, clearly, $(N_1\times N_2,\circ,(0,0))$ is an Almost (m, m+1)-semilattice with zero under the pointwise operation, where $(x_1, y_1)\circ(x_2, y_2) = (x_1\circ x_2, y_1\circ y_2)$ for any $(x_1,y_1), (x_2,y_2) \in N_1\times N_2$.
Then, $f((x_1,y_1)\circ(x_2,y_2)) = f(x_1\circ x_2, y_1\circ y_2)$
$= (x_1\circ x_2, 0)$
$= (x_1\circ x_2, y_1\circ 0)$
$= (x_1,y_1)\circ(x_2,0)$
$= (x_1,y_1)\circ f(x_2,y_2)$
Therefore, f is a multiplier of $N_1\times N_2$.

4. Conclusions

In this paper, we discussed the new concept and defined the conditions in an almost (m, m+1)-

semilattice. Also, we discussed some examples, remarks and theorems based on it.

Author Statements:

- **Ethical approval:** The conducted research is not related to either human or animal use.
- **Conflict of interest:** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper
- **Acknowledgement:** The authors declare that they have nobody or no-company to acknowledge.
- **Author contributions:** The authors declare that they have equal right on this paper.
- **Funding information:** The authors declare that there is no funding to be acknowledged.
- **Data availability statement:** The data that support the findings of this study are available on request from the corresponding author. The data are not publicly available due to privacy or ethical restrictions.

References

[1] Beyene, T. G., & Alemu, H. T. (2011). *Multipliers in almost semilattices*. Southeast Asian Bulletin of Mathematics, 35, 1–9.

[2] A. H. Clifford, G. B. Preston, *The Algebraic Theory of Semigroups*, American Mathematical Society, United States of America, (1964).

[3] Garrett Birkhoff, *Lattice Theory*, American Mathematical Society, Volume XXV.

[4] Geoferlin Jesho, E. A., & Pramila Inpa Rose, A. (2025). Almost (m, n)-semilattice. *International Journal of Applied Mathematics*, *38*(1s), 1461–1467. https://doi.org/10.12732/ijam.v38i1s.757

[5] George Gratzer, *General Lattice Theory*, Academic Press, New York, 1978.

[6] Ivan Chajda, Radomir Halas, Jankuhr, *Semilattice Structures*, Research and Exposition in Mathematics, Vol. 30.

[7] Maddana Swamy.U and Rao.G.C: *Almost Distributive Lattice*, J.Austral. Math. Soc.(Series 31(1981), 77-91.

[8] G.Nanaji Rao and T.G.Beyene, *Almost semilattices*, International Journal of Mathematical Archive 7(3) (2016) 52-67.

[9] Beyene, T. G., & Alemu, H. T. (2011). *Multipliers in almost semilattices*. Southeast Asian Bulletin of Mathematics, 35, 1–9.

[10] M. Velrajan, A. Pramila Inpa Rose, *(m, n)-semilattice*, International Journal of Pure and Applied Mathematics, Volume 119, No.16, 2018, 5077-5092.