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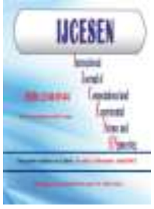

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Research Article

Exact Multiplicity of Positive Solutions for a Quasilinear Boundary Value Problem with Density-Dependent Diffusion and Bistable Nonlinearity

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Abstract:

We investigate the exact number of positive solutions for a quasilinear Dirichlet problem with a density-dependent diffusion coefficient of the form $\varphi_p(u^\alpha)\varphi_p(u')$ and a bistable nonlinearity $\lambda(u^{p-1}-u^{2p-2})(u^{p-1}-c)$, where $\varphi_p(y)=|y|^{p-2}y$, $p>1$, $\alpha>0$, $c>0$, $\lambda>0$. Using the quadrature (time-map) method, we determine the exact multiplicity of positive solutions for all $p>1$. For $1<p\leq 2$, we identify two critical values $\lambda_2<\lambda_1$ such that the problem admits zero, one, or exactly two solutions depending on λ. The paper also includes a physical interpretation of the parameters and a numerical illustration of the theoretical results for $p>2$.

1. Introduction

The purpose of this paper is to study the exact number of positive solutions of the following quasilinear boundary value problem

$$\begin{cases} -\left(\varphi_p(u^\alpha)\varphi_p(u')\right)' = \lambda f(u) & \text{in } (0,1), \\ u>0 \text{ in } (0,1), \\ u(0)=u(1)=0, \end{cases} \qquad (1)$$

where $\varphi_p(y)=|y|^{p-2}y, y\in\mathbb{R}, \alpha>0, p>1$, $c>0, f(u)=(u^{p-1}-u^{2p-2})(u^{p-1}-c)$ and $\lambda>0$.

Problems of type (problem 1) with $p=2$ frequently arise in mathematical models, particularly in physics and fluid mechanics. They describe phenomena such as diffusion, heat conduction, or flows through porous media. In mathematical biology, they are also used to model population dynamics, the spread of chemical substances in biological tissues, or biochemical reactions.

When $\alpha>0$, this model represents a substance in which particles exhibit minimal movement when their numbers are low, but their diffusion velocity increases as the particle density within a given space grows. This behaviour can be used, for instance, to model flows through porous media.

The case $\alpha<0$ describes a scenario where particles move at a very high velocity when their numbers are low, but their speed significantly decreases as their density increases. This models an effect of stickiness or increased resistance to movement at higher particle concentrations (see [9] , p.89).

The present work extends and complements several results from the existing literature. In [1], Addou and Benmezai studied the one-dimensional p-Laplacian boundary value problem

$$\begin{cases} -\left(\varphi_p(u')\right)' & = g(u) & \text{in } (0,1), \\ u & >0 & \text{in } (0,1), \\ u(0)=u(1) & =0 \end{cases}$$

with even superlinear nonlinearities, and established exact multiplicity results via the time-mapping method. Their analysis, however, is restricted to the standard p-Laplacian operator and does not account for the presence of a density-dependent diffusion coefficient of the form $\varphi_p(u^\alpha)$. In contrast, the operator considered in the present paper, namely $\varphi_p(u^\alpha)\varphi_p(u')$, introduces density-dependent diffusion depending on the solution itself, which significantly complicates the analysis of the time-map and requires a more delicate study of its monotonicity properties.

On the other hand, in our previous work [5], we investigated a related quasilinear problem with a p-convex nonlinearity and established exact multiplicity results for positive solutions. While the structural approach in [5] is similar in spirit to ours, the nonlinearity considered there differs substantially from the one studied in the present paper, namely $\lambda(u^{p-1} - u^{2p-2})(u^{p-1} - c)$, which possesses a more complex zero structure and gives rise to a richer bifurcation diagram. In particular, the interplay between the parameters λ, ρ, c and p produces a transition in the solution structure from no solution to exactly two solutions that was not captured in .To the best of our knowledge, the exact number of positive solutions for problem (1) combining the weighted operator $\varphi_p(u^\alpha)\varphi_p(u')$ with the nonlinearity $\lambda(u^{p-1} - u^{2p-2})(u^{p-1} - c)$ has not been previously studied in the literature. The main novelty of this work is therefore twofold: first, we treat a genuinely new combination of a nonlinear diffusion operator and a nonlinearity with a nontrivial zero structure; second, our analysis covers all values $p > 1$ and $\alpha > 0$, unifying the cases $1 < p \leq 2$ and $p > 2$ under a single framework based on the behavior of the time-map $\tilde{T}(p, \lambda, c, \rho)$. We also include a physical interpretation of the parameters and a numerical illustration of the theoretical results for $p > 2$.

This paper is organized as follows. In Section 2 we state the main results. In Section 3 we present the time-mapping approach. Section 4 contains some preliminary lemmas. Section 5 provides the proof of our main results and Section 6 contains a physical interpretation of the parameters and a numerical illustration of the theoretical results for $p > 2$.

2. Mains results

We consider the following problem

$$\begin{cases} -\left(\varphi_p(u^\alpha)\varphi_p(u')\right)' = \lambda f(u) & \text{in } (0,1), \\ u > 0 \text{ in } (0,1), & \\ u(0) = u(1) = 0, & \end{cases} \tag{1}$$

where $\varphi_p(y) = |y|^{p-2}y, y \in \mathbb{R}, \alpha > 0, p > 1$, $c > 0, f(u) = (u^{p-1} - u^{2p-2})(u^{p-1} - c)$ and $\lambda > 0$.

To state our result, define

$$S^+ = \{u \in C^1([0,1]); u > 0 \text{ in } (0,1), u(0) = u(1) = 0 \text{ and } u'(0) > 0\}$$

Let A^+ be the subset of S^+ consisting of the functions u satisfying:

i. u is symmetric about $\frac{1}{2}$.

ii. The derivative of u vanishes exactly once in (0,1).

Let B^+ be the subset of $C^1([0,1])$ composed by the functions u satisfying:

iii. $u > 0$ in $(0,1)$ and $u(0) = u(1) = u'(0) = 0$.

iv. u is symmetric about $\frac{1}{2}$.

v. The derivative of u vanishes exactly once in (0,1).

Solutions in A^+ correspond to symmetric solutions with a single inflection point and a strictly positive initial slope. They are obtained when the energy parameter $E = u'(0) > 0$. Solutions in B^+ correspond to the limiting case $E = 0$, i.e., the derivative vanishes at the origin. This case is important because it occurs at certain critical values of the parameter λ and allows bifurcations to occur. The distinction between A^+ and B^+ is classical in the study of multiplicity via the time- map method.

The main result of this work is:

Theorem 1. *Assume that* $p > 1, 0 < c < \min\left(\frac{p+\alpha}{3p+\alpha-2}, \frac{(\alpha+1)p+\alpha+2}{3(\alpha+1)p+4+\alpha}\right)$ *and* $\alpha > 0$.

(A) *If* $1 < p \leq 2$, *then there exist* $\lambda_1(p, \alpha, c) > 0$ *and* $\lambda_2(p, \alpha, c)$ *such that*

i. *If* $\lambda > \lambda_1(p, \alpha, c)$, *then the problem* (problem 1) *admits a unique solution in* A^+,

ii. *If* $\lambda = \lambda_1(p,\alpha,c)$, *then the problem* (problem 1) *admits two solutions, one in* A^+ *and the other one in* B^+,

iii. *If* $\lambda < \lambda_2(p,\alpha,c)$, *then the problem* (problem 1) *admits no solution,*

iv. *If* $\lambda_2(p,\alpha,c) < \lambda < \lambda_1(p,\alpha,c)$, *then the problem* (problem 1) *admits exactly two solutions in* A^+,

v. *If* $\lambda = \lambda_2(p,\alpha,c)$, *then the problem* (problem 1) *admits a unique solution in* B^+.

(B) *If* $p > 2$, *then there exists* $\lambda_3(c,p,\alpha)$, $\lambda_*(c,p,\alpha)$, $\lambda_{**}(c,p,\alpha)$ *and* $\tilde{\alpha}$ *such that*

i. *If* $\lambda < \lambda_2(p,\alpha,c)$, *then the problem* (problem 1) *admits no solution,*

ii. *If* $\lambda = \lambda_2(p,\alpha,c)$, *then the problem* (problem 1) *admits a unique solution in* A^+,

iii. *If* $\lambda > \lambda_3(c,p,\alpha)$ *and* $\alpha > \tilde{\alpha}$, *then the problem* (problem 1) *admits a unique solution in* A^+,

iv. *if* $\lambda > \max(\lambda_*(c,p,\alpha), \lambda_{**}(c,p,\alpha))$, *then the problem* (problem 1) *admits exactly two solutions in* A^+.

3. Time-mapping approach

In this section we introduce the well-known time mapping approach (see for instance).

Consider the following boundary value problem:

$$\begin{cases} -\left(\varphi_p(u')\right)' = g(u) \text{ in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \qquad (2)$$

where $g \in C(\mathbb{R}_+, \mathbb{R})$.

Define $G(s) := \int_0^s g(t)dt$.

For any $E \geq 0$ and $p > 1$, let

$$X_+(p,E) = \left\{ s > 0;\ E^p - \frac{p}{p-1} G(\zeta) > 0, \forall \zeta,\ 0 < \zeta < s \right\}$$

and

$$S_+(p,E) = \begin{cases} 0 \text{ if } X_+(p,E) = \emptyset \\ \text{Sup} X_+(p,E) \text{ otherwise} \end{cases}$$

Let

$$D = \{E \geq 0;\ 0 < S_+(p,E) < +\infty \text{ and } g(S_+(p,E)) > 0\}.$$

We define the following time-map

$$T_+(p,E) = \int_0^{S_+(p,E)} \left[E^p - \frac{p}{p-1} G(u) \right]^{-\frac{1}{p}} du.$$

We now state the following well-known theorem without proof (see for instance [6]).

Theorem 2. *Assume that* $g \in C(\mathbb{R}_+, \mathbb{R})$, $E \geq 0$ *and* $p > 1$. *Then:*

- *Problem (2) admits a solution* $u \in A^+$ *satisfying* $u'(0) = E$ *if and only if* $E \in D \cap (0, +\infty)$ *and* $T_+(p,E) = \frac{1}{2}$. *In this case the solution is unique and its sup-norm is equal to* $S_+(p,E)$.
- *Problem (2) admits a solution* $u \in B^+$ *if and only if* $0 \in D$ *and* $T_+(p,0) = \frac{1}{2}$. *In this case the solution is unique and its sup-norm is equal to* $S_+(p,0)$.

Now we consider the following problem

$$\begin{cases} -\left(\varphi_p(u^\alpha)\varphi_p(u')\right)' = \lambda f(u) \qquad \text{in } (0,1), \\ u > 0 \text{ in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \qquad (3)$$

where $\varphi_p(y) = |y|^{p-2}y, y \in \mathbb{R}, \alpha > 0, p > 1$, $c > 0, f(u) = (u^{p-1} - u^{2p-2})(u^{p-1} - c)$, $0 < c < \dfrac{p+\alpha}{3p+\alpha-2}$ and $\lambda > 0$.

If we put $v = \frac{u^{\alpha+1}}{\alpha+1}$, we obtain

$$\begin{cases} -\left(\varphi_p(v')\right)' = \lambda h(v) \text{in } (0,1), \\ v > 0 \text{ in } (0,1), \\ v(0) = v(1) = 0, \end{cases}$$

where

$$h(v) = (\alpha+1)^{\frac{p-1}{\alpha+1}} v^{\frac{p-1}{\alpha+1}} \left[(\alpha+1)^{\frac{p-1}{\alpha+1}} v^{\frac{p-1}{\alpha+1}} \left(1 + c - (\alpha+1)^{\frac{p-1}{\alpha+1}} v^{\frac{p-1}{\alpha+1}} \right) - c \right].$$

4. Preliminary lemmas

Lemma 1. *Consider the equation in* $s \in R_+$:

$$E^p - \frac{p}{p-1}\lambda H(s) = 0, \quad (4)$$

where $E \geq 0, p > 1, 0 < c < \frac{p+\alpha}{3p+\alpha-2}, \lambda > 0$ and $H(s) = \int_0^s h(t)dt$, for all $s \geq 0$.

Then there exists $E_* = \left(\frac{\lambda p}{p-1} H\left(\frac{1}{\alpha+1}\right)\right)^{\frac{1}{p}}$ *such that for all* $E \in (0, E_*)$ *equation (4) admits a unique positive zero* $r_+(p,\lambda,c,E)$*. Moreover*

i. *The function* $E \mapsto r_+(p,\lambda,c,E)$ *is* C^1 *in* $(0,+\infty)$ *and for all* $p > 1$, $\lambda > 0$, $c > 0$ *and* $E > 0$, *we have* $\frac{\partial r_+}{\partial E}(p,\lambda,c,E) = \frac{(p-1)E^{p-1}}{\lambda h(r_+(p,\lambda,c,E))}$.

ii. $\lim_{E\to 0^+} r_+(p,\lambda,c,E) = \frac{(\widetilde{w})^{\frac{p-1}{\alpha+1}}}{\alpha+1}$, *where* $\widetilde{w}$ *is the unique positive root of* $h(v) = 0$ *in* $\left(0, \frac{1}{\alpha+1}\right)$ *(after the change of variable* $w = (\alpha+1)^{\frac{p-1}{\alpha+1}} v^{\frac{p-1}{\alpha+1}}$*). It is given by*
$$\widetilde{w} = \frac{3p+\alpha-2}{2}\left(\frac{1+c}{2p+\alpha-1} - \sqrt{\frac{(1+c)^2}{(2p+\alpha-1)^2} - \frac{4c}{(3p+\alpha-2)(p+\alpha)}}\right).$$

iii. $\lim_{E\to E_*} r_+(p,\lambda,c,E) = \frac{1}{\alpha+1}$.

Proof. The proof of this lemma is similar to that of Lemma 4.1 in or Lemma 8 in . So, it is omitted. □

Now, we are ready, for any $p > 1, \lambda > 0, 0 < c < \frac{p+\alpha}{3p+\alpha-2}$ and $E \geq 0$, to compute $X_+(p,\lambda,c,E)$ as defined in Section 3.
In fact

$$X_+(p,\lambda,c,E) =]0, r_+(p,\lambda,c,E)[.$$

Then

$$S_+(p,\lambda,E) = r_+(p,\lambda,c,E).$$

On the other hand, we have

$$D = \{E \geq 0, 0 < s_+(p,\lambda,c,E) < +\infty \text{ and } h(s_+(p,\lambda,c,E)) > 0\} = [0, E_*[.$$

By lemma 1, we have

$$\lim_{E\to 0^+} s_+(p,\lambda,c,E) = \frac{(\widetilde{w})^{\frac{p-1}{\alpha+1}}}{\alpha+1},$$

$$\lim_{E\to +E_*} s_+(p,\lambda,c,E) = \frac{1}{\alpha+1},$$

and

$$\frac{\partial s_+}{\partial E}(p,\lambda,c,E) = \frac{(p-1)E^{p-1}}{\lambda f(s_+(p,\lambda,c,E)} > 0, \forall p > 1,$$

$$\forall \lambda > 0, \forall c > 0, \text{ and } \forall E \geq 0.$$

At present, we define, for any $p > 1, \lambda > 0, 0 < c < \frac{p+\alpha}{3p+\alpha-2}$ and $0 \leq E \leq E_*$, the time-map T_+ by

$$T_+(p,\lambda,c,E) = \int_0^{s_+(p,\lambda,c,E)} \left[E^p - \frac{p}{p-1}\lambda H(s)\right]^{-\frac{1}{p}} du. \quad (5)$$

Now if we put the change of variables $u = s_+(p,\lambda,E)t$ in (5), we obtain that

$$T_+(p,\lambda,c,E) = \left(\frac{p}{p-1}\lambda\right)^{-\frac{1}{p}} \int_0^1 \left[H(s_+(p,\lambda,c,E)) - H(s_+(p,\lambda,c,E)t)\right]^{-\frac{1}{p}} dt.$$

We observe that

$$T_+(p,\lambda,c,E) = \tilde{T}(p,\lambda,s_+(p,\lambda,c,E)), \text{ for all } p > 1, \lambda > 0 \text{ and } E \geq 0,$$

where

$$\tilde{T}(p,\lambda,c,\rho) = \left(\frac{p}{p-1}\lambda\right)^{-\frac{1}{p}} \int_0^1 [H(\rho) - H(\rho t)]^{-\frac{1}{p}} dt, \quad (6)$$

for all $p > 1, \lambda > 0$, $0 < c < \frac{p+\alpha}{3p+\alpha-2}$ and $\rho > 0$.
Since the function $E \mapsto s_+(p,\lambda,c,E)$ is an increasing C^1 −diffeomorphism from $(0,+\infty)$ onto itself it follows that if we put, for all $p > 1$ and $\lambda > 0$, we have

$$J_1(p,\lambda,c) := \left\{E \geq 0 : T_+(p,\lambda,c,E) = \frac{1}{2}\right\},$$

and

$$J_2(p,\lambda,c) := \left\{\rho > 0 : \tilde{T}(p,\lambda,c,\rho) = \frac{1}{2}\right\},$$

then

$$\text{Card}\left(J_1(p,\lambda,c)\right) = \text{Card}\left(J_2(p,\lambda,c)\right), \text{ for all } p > 1, c > 0 \text{ and } \lambda > 0.$$

Hence, from now, we will focus our attention in counting the number of solutions of the equation $\tilde{T}(p,\lambda,c,\rho)=\frac{1}{2}$ in the variable $\rho>0$, instead of the equation $T_{+}(p,\lambda,c,E)=\frac{1}{2}$ in the variable $E>0$.

Proposition 3. *If u is a positive solution of problem ([problem01]), then $u\in A^{+}\cup B^{+}$.*

Proof. The change of variable $v=u^{\alpha+1}/(\alpha+1)$ transforms ([problem01]) into a standard p- Laplacian problem

$$-\left(\varphi_{p}(v')\right)'=\lambda h(v),\quad v(0)=v(1)=0,\ v>0,$$

where h is given in Section 3. For such a problem, it is known (see e.g.) that any positive solution is symmetric about $x=1/2$ and satisfies either $v'(0)>0$ or $v'(0)=0$. Translating back to u, this means $u\in A^{+}$ when $u'(0)>0$ and $u\in B^{+}$ when $u'(0)=0$. Hence the result. ☐

Lemma 2. *For all $p>1$, $\alpha>0$, $0<c<\frac{p+\alpha}{3p+\alpha-2}$ and $\lambda>0$, we have*

1. $$\lim_{\rho\to\rho_{*}}\tilde{T}(p,\lambda,c,\rho)=M(c,p,\alpha),$$

where

$$\rho_{*}=\frac{\tilde{w}^{\frac{p-1}{\alpha+1}}}{\alpha+1},$$

$$AndM(c,p,\alpha)=\left(\lambda\frac{p-1}{p}\right)^{-\frac{1}{p}}\frac{\tilde{w}^{\frac{p-1}{\alpha+1}}}{\alpha+1}\tilde{w}^{\frac{-(p+\alpha)}{p}}\int_{0}^{1}\left[\frac{-\tilde{w}^{2}}{3p+\alpha-2}\left(1t^{\frac{3(p-1)}{\alpha+1}+1}\right)+\tilde{w}\left(\frac{1+c}{2p+\alpha-1}\right)\left(1-t^{\frac{2(p-1)}{\alpha+1}+1}\right)-\left(\frac{c}{p+\alpha}\right)\left(1-t^{\frac{p-1}{\alpha+1}+1}\right)\right]^{-\frac{1}{p}}dt.$$

2. $$\lim_{\rho\to+\frac{1}{\alpha+1}}\tilde{T}(p,\lambda,c,\rho)=\begin{cases}M_{1}\left(\frac{1}{\alpha+1},p\right) & \text{if } p>2,\\ +\infty & \text{otherwise,}\end{cases}$$

Where

$$M_{1}\left(\frac{1}{\alpha+1},p\right)=\left(\frac{\lambda p}{p-1}\right)^{-\frac{1}{p}}\int_{0}^{\frac{1}{\alpha+1}}\left[H\left(\frac{1}{\alpha+1}\right)-H(u)\right]^{-\frac{1}{p}}du$$

Proof. Let $p>1$, $0<c<\frac{p+\alpha}{3p+\alpha-2}$, $\alpha>0$ and $\lambda>0$ be fixed.

1. We have

$$\lim_{\rho\to\frac{\tilde{w}^{\frac{p-1}{\alpha+1}}}{\alpha+1}}\tilde{T}(p,\lambda,c,\rho)=\lim_{\rho\to\frac{\tilde{w}^{\frac{p-1}{\alpha+1}}}{\alpha+1}}\left(\frac{p-1}{p}\right)^{\frac{-1}{p}}\int_{0}^{1}\rho\left[H(\rho)-H(\rho t)\right]^{\frac{-1}{p}}dt.$$

If we put $w=(\alpha+1)^{\frac{p-1}{\alpha+1}}\rho^{\frac{p-1}{\alpha+1}}$, we obtain

$$H(w)-H(wt)=w^{p+\alpha}\left[\frac{-w^{2}}{3p+\alpha-2}\left(1-t^{\frac{3(p-1)}{\alpha+1}+1}\right)+w\left(\frac{1+c}{2p+\alpha-1}\right)\left(1-t^{\frac{2(p-1)}{\alpha+1}+1}\right)-\left(\frac{c}{p+\alpha}\right)\left(1-t^{\frac{p-1}{\alpha+1}+1}\right)\right],$$

Then, we have

$$\lim_{\rho\to\rho_{*}}\tilde{T}(p,\lambda,c,\rho)=\left(\frac{p-1}{p}\right)^{\frac{-1}{p}}\frac{\tilde{w}^{\frac{p-1}{\alpha+1}}}{\alpha+1}\tilde{w}^{\frac{-(p+\alpha)}{p}}\int_{0}^{1}[K(\tilde{w})]^{\frac{-1}{p}}dt,$$

where

$$K(\tilde{w})=\frac{-\tilde{w}^{2}}{3p+\alpha-2}\left(1-t^{\frac{3(p-1)}{\alpha+1}+1}\right)+\tilde{w}\left(\frac{1+c}{2p+\alpha-1}\right)\left(1-t^{\frac{2(p-1)}{\alpha+1}+1}\right)-\left(\frac{c}{p+\alpha}\right)\left(1-t^{\frac{p-1}{\alpha+1}+1}\right).$$

2. We have

$$\lim_{\rho\to\frac{1}{\alpha+1}}\tilde{T}(p,\lambda,c,\rho)=\lim_{\rho\to\frac{1}{\alpha+1}}\int_{0}^{\rho}[H(\rho)-H(u)]^{-\frac{1}{p}}du$$

$$=\left(\frac{\lambda p}{p-1}\right)^{-\frac{1}{p}}\int_{0}^{\frac{1}{\alpha+1}}\left[H\left(\frac{1}{\alpha+1}\right)-H(u)\right]^{-\frac{1}{p}}du$$

$$=\begin{cases}M_{1}\left(\frac{1}{\alpha+1},p\right) & \text{if } p>2,\\ +\infty & \text{otherwise.}\end{cases}$$

☐

Lemma 3. *For all* $p > 1$, $0 < c < \frac{p+\alpha}{3p+\alpha-2}$, *one has*

i. $\lim\limits_{\alpha\to+\infty} M(c,p,\alpha) = 0.$

ii. $\lim\limits_{\alpha\to+\infty} M_1(c,p,\alpha) = +\infty.$

Proof. Let $p > 1$, $0 < c < \frac{p+\alpha}{3p+\alpha-2}$, $\alpha > 0$ and $\lambda > 0$ be fixed.

i. We have

$$M(c,p,\alpha) =$$

$$\left(\lambda\frac{p-1}{p}\right)^{-\frac{1}{p}}\frac{\widetilde{w}^{\frac{p-1}{\alpha+1}}}{\alpha+1}\widetilde{w}^{\frac{-(p+\alpha)}{p}}\int_0^1\left[\frac{-\widetilde{w}^2}{3p+\alpha-2}\left(1-\right.\right.$$

$$\left.t^{\frac{3(p-1)}{\alpha+1}+1}\right)+\widetilde{w}\left(\frac{1+c}{2p+\alpha-1}\right)\left(1-\right.$$

$$\left.\left.t^{\frac{2(p-1)}{\alpha+1}+1}\right)-\left(\frac{c}{p+\alpha}\right)\left(1-t^{\frac{p-1}{\alpha+1}+1}\right)\right]^{-\frac{1}{p}}dt.$$

Since

$$\widetilde{w} =$$

$$\frac{3p+\alpha-2}{2}\left[\frac{1+c}{2p+\alpha-1}-\sqrt{\frac{(1+c)^2}{(2p+\alpha-1)^2}-\frac{4c}{(3p+\alpha-2)(p+\alpha)}}\right]$$

Then

$$\lim_{\alpha\to+\infty}\widetilde{w} = \frac{1+c}{2}-\frac{1}{2}\sqrt{(1-c)^2} = c,$$

and consequently

$$\lim_{\alpha\to+\infty} M(c,p,\alpha) = 0.$$

ii. We have

$$M_1\left(\frac{1}{\alpha+1},p\right) = \left(\frac{\lambda p}{p-1}\right)^{-\frac{1}{p}}\int_0^1\left[H\left(\frac{1}{\alpha+1}\right)-\right.$$

$$\left.H(u)\right]^{-\frac{1}{p}}du.$$

If we put the change of variables $u = \frac{1}{\alpha+1}\tau$, we obtain

$$\int_0^1\left[H\left(\frac{1}{\alpha+1}\right)-H\left(\frac{1}{\alpha+1}\tau\right)\right]^{-\frac{1}{p}}d\tau$$

$$= \left(\lambda\frac{p-1}{p}\right)^{-\frac{1}{p}}(\alpha+1)$$

$$\times\int_0^1\left[\frac{1+c}{2p+\alpha-1}\left(1-\tau^{\frac{2(p-1)}{\alpha+1}+1}\right)\right.$$

$$-\frac{1}{3p+\alpha-2}\left(1-\tau^{\frac{3(p-1)}{\alpha+1}+1}\right)-$$

$$\left.\frac{c}{p+\alpha}\left(1-\tau^{\frac{p-1}{\alpha+1}+1}\right)\right]^{-\frac{1}{p}}d\tau$$

$$= \left(\lambda\frac{p-1}{p}\right)^{-\frac{1}{p}}$$

$$\times\int_0^1\left[\frac{(\alpha+1)^{-p}}{\frac{1+c}{2p+\alpha-1}\left(1-\tau^{\frac{2(p-1)}{\alpha+1}+1}\right)-}\frac{1}{3p+\alpha-2}\left(1\right.\right.$$

$$\left.-\tau^{\frac{3(p-1)}{\alpha+1}+1}\right)\frac{c}{p+\alpha}$$

$$\left.\left(1-\tau^{\frac{p-1}{\alpha+1}+1}\right)\right]^{-\frac{1}{p}}d\tau.$$

Which implies that

$$\lim_{\alpha\to+\infty} M_1\left(\frac{1}{\alpha+1},p\right) = +\infty.$$

Lemma 4. *For all* $p > 1, 0 < c <$ $\min\left(\frac{p+\alpha}{3p+\alpha-2},\frac{(\alpha+1)p+\alpha+2}{3(\alpha+1)p+4+\alpha}\right)$, $\alpha > 0$ *and* $\lambda > 0$, *we have*

i. *There exist* $\rho_1 > 0$ *such that the function* $\rho \mapsto \tilde{T}(p,\lambda,c,\rho)$ *is strictly decreasing on* (ρ_*,ρ_1).

ii. *There exist* $\rho_2 > \rho_1$ *such that the function* $\rho \mapsto \tilde{T}(p,\lambda,c,\rho)$ *is strictly increasing on* $\left(\rho_2,\frac{1}{\alpha+1}\right)$.

Proof. Let $p > 1$, $\alpha > 0$ and $\lambda > 0$ be fixed.

Differentiating (6) with respect to ρ, we obtain

$$\frac{\partial\tilde{T}}{\partial\rho}(p,\lambda,c,\rho)$$

$$= \frac{1}{p\rho}\left(\frac{\lambda p}{p-1}\right)^{-\frac{1}{p}}\int_0^\rho\frac{L(\rho)-L(u)}{[H(\rho)-H(u)]^{\frac{p+1}{p}}}du,$$

where

$$L(u) = pH(u)-uh(u).$$

If we put the change of variables $w = (\alpha+1)^{\frac{p-1}{\alpha+1}}u^{\frac{p-1}{\alpha+1}}$, we obtain

$$\tilde{L}(w)$$

$$= \left(\frac{\alpha+1}{p-1}\right)w^{\frac{\alpha+1}{p-1}+1}\left[\left(\frac{2-\alpha}{3p+\alpha-2}\right)w^2+\frac{(1+c)(\alpha-1)}{2p+\alpha-1}w\right.$$

$$\left.-\frac{c\alpha}{p+\alpha}\right],$$

with

$$\tilde{L}(w) = L(u).$$

Some easy computations show that

$$\tilde{L}(1) = p[-c[3(\alpha+1)p + (4+\alpha)] + (\alpha+1)p + (\alpha+2)],$$

and consequently, it follows that

$$\tilde{L}(1) > 0 \text{ if } c < \frac{(\alpha+1)p+\alpha+2}{3(\alpha+1)p+4+\alpha}.$$

On the other hand, we have

$$\tilde{L}'(w) = \frac{w^{\alpha+1}}{\alpha+1}[(2-\alpha)w^2 + (1+c)(\alpha-1)w - c\alpha].$$

To study the variations of the function $\tilde{L}'(w)$ it suffices to study those of the function G defined by

$$G(w) = (2-\alpha)w^2 + (1+c)(\alpha-1)w - c\alpha.$$

We have

$$G(0) = -c\alpha < 0\,,$$

$$G(1) = 1 - c > 0,$$

and

$$G'(w) = 2(2-\alpha)w + (1+c)(\alpha-1).$$

We distinguish four cases

First case: $\alpha \in [0,1[$

In this case, there exist two real numbers $\tilde{\rho}_1$and $\tilde{\rho}_2$ with $0 < \tilde{\rho}_2 < \tilde{\rho}_1$ such that

$$G(w) < 0 \text{ on } (0,\tilde{\rho}_2), G(\tilde{\rho}_2) = 0,$$

and

$$G(w) > 0 \text{ on } (\tilde{\rho}_2, 1).$$

So, we have

$$\tilde{L}'(w) < 0 \text{ on } (0,\tilde{\rho}_2),\ \tilde{L}(\tilde{\rho}_2) = 0,$$

and

$$\tilde{L}'(w) > 0 \text{ on } (\tilde{\rho}_2, 1),$$

and since $\tilde{L}(1) > 0$, it follows that

$$\tilde{L}(w) < 0 \text{ on } (0,\tilde{\rho}_1), \tilde{L}(\tilde{\rho}_1) = 0 \text{ and } \tilde{L}(w) > 0 \text{ on } (\tilde{\rho}_1, 1).$$

Second case: $\alpha \in [1,2[$

In this case, we have

$$G(w) < 0 \text{ on } (0,\tilde{\rho}_2), G(\tilde{\rho}_2) = 0,$$

and

$$G(w) > 0 \text{ on } (\tilde{\rho}_2, 1).$$

Then, we have

$$\tilde{L}'(w) < 0 \text{ on } (0,\tilde{\rho}_2), \tilde{L}(\tilde{\rho}_2) = 0,$$

and

$$\tilde{L}'(w) > 0 \text{ on } (\tilde{\rho}_2, 1),$$

and consequently, it follows that

$$\tilde{L}(w) < 0 \text{ on } (0,\tilde{\rho}_1), \tilde{L}(\tilde{\rho}_1) = 0,$$

and

$$\tilde{L}(w) > 0 \text{ on } (\tilde{\rho}_1, 1).$$

Third case: $\alpha = 2$

In this case, we have

$$G(w) < 0 \text{ on } (0,\tilde{\rho}_2), G(\tilde{\rho}_2) = 0,$$

and

$$G(w) > 0 \text{ on } (\tilde{\rho}_2, 1),$$

Then

$$\tilde{L}'(w) < 0 \text{ on } (0,\tilde{\rho}_2),$$

$$\tilde{L}(\tilde{\rho}_2) = 0,$$

and

$$\tilde{L}'(w) > 0 \text{ on } (\tilde{\rho}_2, 1),$$

and consequently

$$\tilde{L}(w) < 0 \text{ on } (0,\tilde{\rho}_1), \tilde{L}(\tilde{\rho}_1) = 0,$$

and

$$\tilde{L}(w) > 0 \text{ on } (\tilde{\rho}_1, 1).$$

Fourth case: $\alpha > 2$

We have two subcases

First subcase: $\alpha \in \left]2, \frac{3}{1-c}\right]$.

In this case, we have

$$G(w) < 0 \text{ on } (0,\tilde{\rho}_2), G(\tilde{\rho}_2) = 0,$$

and

$$G(w) > 0 \text{ on } (\tilde{\rho}_2, 1),$$

Then, we have

$$\tilde{L}'(w) < 0 \text{ on } (0, \tilde{\rho}_2), \tilde{L}(\tilde{\rho}_2) = 0,$$

and

$$\tilde{L}'(w) > 0 \text{ on } (\tilde{\rho}_2, 1),$$

and consequently, it follows that

$$\tilde{L}(w) < 0 \text{ on } (0, \tilde{\rho}_1), \tilde{L}(\tilde{\rho}_1) = 0,$$

and

$$\tilde{L}(w) > 0 \text{ on } (\tilde{\rho}_1, 1).$$

Second subcase: $\alpha \in \left]\frac{3}{1-c}, +\infty\right[$.
In this case, there exists $\tilde{\rho}_2 < \rho_* < \tilde{\rho}_1$ such that

$$G'(w) > 0 \text{ on } (0, \rho_*), G'(\rho_*) = 0,$$

and

$$G'(w) < 0 \text{ on } (\rho_*, 1).$$

Which implies that

$$G(w) < 0 \text{ on } (0, \tilde{\rho}_2), G(\tilde{\rho}_2) = 0,$$

and

$$G(w) > 0 \text{ on } (\tilde{\rho}_2, 1).$$

Then, we have

$$\tilde{L}'(w) < 0 \text{ on } (0, \tilde{\rho}_2), \tilde{L}(\tilde{\rho}_2) = 0,$$

and

$$\tilde{L}'(w) > 0 \text{ on } (\tilde{\rho}_2, 1),$$

and consequently, it follows that

$$\tilde{L}(w) < 0 \text{ on } (0, \tilde{\rho}_1), \tilde{L}(\tilde{\rho}_1) = 0,$$

and

$$\tilde{L}(w) > 0 \text{ on } (\tilde{\rho}_1, 1).$$

So in all cases, we have

$$\frac{\partial \tilde{T}}{\partial \rho}(p, \lambda, c, \rho) < 0 \text{ for all } p > 1, \lambda > 0, \alpha > 0 \text{ and } \rho \in (0, \rho_2),$$

and

$$\frac{\partial \tilde{T}}{\partial \rho}(p, \lambda, c, \rho) > 0 \text{ for } p > 1, \lambda > 0, \alpha > 0 \text{ and } \rho \in \left(\rho_1, \frac{1}{\alpha+1}\right).$$

This means that for all $p > 1$, $\lambda > 0$ and $\alpha > 0$, the function $\rho \mapsto \tilde{T}(p, \lambda, c, \rho)$ is strictly decreasing on (ρ_*, ρ_1) and strictly increasing on $\left(\rho_2, \frac{1}{\alpha+1}\right)$. □

Now we are going to prove that the time-map $\rho \mapsto \tilde{T}(p, \lambda, c, \rho)$ admits a unique critical point on (ρ_1, ρ_2).

Proposition 4. *For all* $p > 1$, $\lambda > 0$ *and* $\alpha > 0$, *if there exist* $\tilde{\rho} \in (\rho_1, \rho_2)$ *such that* $\tilde{T}'(\tilde{\rho}) = 0$, *then* $\tilde{T}''(\tilde{\rho}) > 0$.

Proof. For all $p > 1$, $\lambda > 0$ and $\alpha > 0$, we have

$$\tilde{T}(p, \lambda, c, \rho) = \left(\frac{p}{p-1}\lambda\right)^{-\frac{1}{p}} \int_0^1 [H(\rho) - H(\rho t)]^{-\frac{1}{p}} dt,$$

$$\tilde{T}'(p, \lambda, c, \rho) = \frac{1}{p\rho}\left(\frac{\lambda p}{p-1}\right)^{-\frac{1}{p}} \int_0^\rho \frac{L(\rho) - L(u)}{[H(\rho) - H(u)]^{\frac{p+1}{p}}} dt,$$

and

$$\tilde{T}''(p, \lambda, c, \rho) = \frac{1}{p}\int_0^\rho \frac{(\Delta H)\big(\rho L'(\rho) - uL'(u)\big) - \left(\frac{p+1}{p}\right)\Delta L\big(\Delta\tilde{h}\big)}{\rho^2\big(H(\rho) - H(u)\big)^{\frac{2p+1}{p}}} du,$$

where

$$\Delta H = H(\rho) - H(u),$$

and

$$\Delta\tilde{h} = \rho h(\rho) - uh(u).$$

We have

$$\tilde{T}''(p, \lambda, c, \rho) + \left(\frac{p+1}{\rho}\right)\tilde{T}'(p, \lambda, c, \rho) = \int_0^\rho \frac{\left(\frac{p+1}{p}\right)(\Delta L)^2 + (\Delta H)\big(\Delta\tilde{L}'\big)}{p\rho^2(\Delta H)^{\frac{2p+1}{p}}} du,$$

where

$$\Delta\tilde{L}' = \rho L'(\rho) - uL'(u).$$

Now, we are going to study the sign of $\Delta\tilde{L}'$.We put the change of variables $w = (\alpha + 1)^{\frac{p-1}{\alpha+1}}\rho^{\frac{p-1}{\alpha+1}}$, we obtain

$$\left(w\tilde{L}'(w)\right)' = \frac{w^{\alpha+1}}{\alpha+1}[(2-\alpha)(\alpha+4)w^2 + (1+c)(\alpha-1)(\alpha+3)w - c\alpha(\alpha+2)] .$$

We put

$$\tilde{G}(w) = (2-\alpha)(\alpha+4)w^2 + (1+c)(\alpha-1)(\alpha+3)w - c\alpha(\alpha+2).$$

We have

$$\tilde{G}(0) = -c\alpha(\alpha+2) < 0, \tilde{G}(1) = 5-3c > 0,$$

and

$$\tilde{G}'(w) = (2-\alpha)(\alpha+4)w + (1+c)(\alpha-1)(\alpha+3).$$

The sign of $\left(w\tilde{L}'(w)\right)'$ is the sign of $\tilde{G}(w)$, so we distinguish four cases.

First case: $\alpha \in [0,1[$

In this case, there exists $\tilde{\rho}_3$ with $0 < \tilde{\rho}_3 < \tilde{\rho}_2 < \tilde{\rho}_1$ such that

$$\tilde{G}(w) < 0 \text{ on } (0,\tilde{\rho}_3), \tilde{G}(\tilde{\rho}_3) = 0,$$

and

$$\tilde{G}(w) > 0 \text{ on } (\tilde{\rho}_3, 1).$$

So, we have

$$\left(w\tilde{L}'(w)\right)' < 0 \text{ on } (0,\tilde{\rho}_3), w\tilde{L}'(\tilde{\rho}_3) = 0,$$

and

$$\left(w\tilde{L}'(w)\right)' > 0 \text{ on } (\tilde{\rho}_3, 1),$$

and consequently, it follows that

$$w\tilde{L}'(w) < 0 \text{ on } (0,\tilde{\rho}_2), w\tilde{L}'(\tilde{\rho}_2) = 0,$$

and

$$w\tilde{L}'(w) > 0 \text{ on } (\tilde{\rho}_2, 1).$$

Second case: $\alpha \in [1,2[$

In this case, there exist $\tilde{\rho}_3$ with $0 < \tilde{\rho}_3 < \tilde{\rho}_2 < \tilde{\rho}_1$ such that

$$\tilde{G}(w) < 0 \text{ on } (0,\tilde{\rho}_3), \tilde{G}(\tilde{\rho}_3) = 0,$$

and

$$\tilde{G}(w) > 0 \text{ on } (\tilde{\rho}_3, 1),$$

so, we have

$$\left(w\tilde{L}'(w)\right)' < 0 \text{ on } (0,\tilde{\rho}_3), w\tilde{L}'(\tilde{\rho}_3) = 0,$$

and

$$\left(w\tilde{L}'(w)\right)' > 0 \text{ on } (\tilde{\rho}_3, 1),$$

and consequently, it follows that

$$w\tilde{L}'(w) < 0 \text{ on } (0,\tilde{\rho}_2), w\tilde{L}'(\tilde{\rho}_2) = 0,$$

and

$$w\tilde{L}'(w) > 0 \text{ on } (\tilde{\rho}_2, 1).$$

Third case: $\alpha = 2$.

In this case, there exist $\tilde{\rho}_3$ with $0 < \tilde{\rho}_3 < \tilde{\rho}_2 < \tilde{\rho}_1$ such that

$$\tilde{G}(w) < 0 \text{ on } (0,\tilde{\rho}_3), \tilde{G}(\tilde{\rho}_3) = 0,$$

and

$$\tilde{G}(w) > 0 \text{ on } (\tilde{\rho}_3, 1).$$

so, we have

$$\left(w\tilde{L}'(w)\right)' < 0 \text{ on } (0,\tilde{\rho}_3), w\tilde{L}'(\tilde{\rho}_3) = 0,$$

and

$$\left(w\tilde{L}'(w)\right)' > 0 \text{ on } (\tilde{\rho}_3, 1),$$

and consequently, it follows that

$$w\tilde{L}'(w) < 0 \text{ on } (0,\tilde{\rho}_2), w\tilde{L}'(\tilde{\rho}_2) = 0,$$

and

$$w\tilde{L}'(w) > 0 \text{ on } (\tilde{\rho}_2, 1).$$

Fourth case: $\alpha > 2$.

In this case we prove that the function $\tilde{G}$ is strictly increasing, so there exist $\tilde{\rho}_3$ with $0 < \tilde{\rho}_3 < \tilde{\rho}_2 < \tilde{\rho}_1$ such that

$$\tilde{G}(w) < 0 \text{ on } (0,\tilde{\rho}_3), \tilde{G}(\tilde{\rho}_3) = 0,$$

and

$$\tilde{G}(w) > 0 \text{ on } (\tilde{\rho}_3, 1),$$

so, we have

$$\left(w\tilde{L}'(w)\right)' < 0 \text{ on } (0,\tilde{\rho}_3), w\tilde{L}'(\tilde{\rho}_3) = 0,$$

and

$$\left(w\tilde{L}'(w)\right)' > 0 \text{ on } (\tilde{\rho}_3, 1).$$

Which implies that

$$w\tilde{L}'(w) < 0 \text{ on } (0,\tilde{\rho}_2), w\tilde{L}'(\tilde{\rho}_2) = 0,$$

and

$$w\tilde{L}'(w) > 0 \text{ on } (\tilde{\rho}_2, 1).$$

Then, we have

$$\left(w\tilde{L}'(w)\right)' > 0 \text{ on } (\tilde{\rho}_3, 1).$$

Which implies that

$$\tilde{T}''(p,\lambda,c,\rho) + \left(\frac{p+1}{\rho}\right)\tilde{T}'(p,\lambda,c,\rho) > 0 \text{ on } (\rho_2, \rho_1),$$

and consequently, it follows that if there exist $\tilde{\rho} \in (\rho_2, \rho_1)$ such that $\tilde{T}'(\tilde{\rho}) = 0$, then $\tilde{T}''(\tilde{\rho}) > 0$. □

5. Proof of theorem 1

Assume that $p > 1, \alpha > 0$ and $\lambda > 0$.

Proof of assertion (A)

Assume that $p \in]1,2]$, by lemmas 2 and 4, we have

i. $\lim_{\rho \to \rho_*} \tilde{T}(p,\lambda,c,\rho) = M(c,p,\alpha)$,

ii. $\lim_{\rho \to \frac{1}{\alpha+1}} \tilde{T}(p,\lambda,c,\rho) = +\infty$.

And The function $\rho \mapsto \tilde{T}(p,\lambda,c,\rho)$ is strictly decreasing on (ρ_*, ρ_1) and strictly increasing on $\left(\rho_1, \frac{1}{\alpha+1}\right)$.

So, the equation in the variable ρ, $\rho \mapsto \tilde{T}(p,\lambda,c,\rho) = \frac{1}{2}$ admits a unique solution in $\left(\rho_*, \frac{1}{\alpha+1}\right)$ if and only if $\lambda > \lambda_1(c,p,\alpha)$ or $\lambda = \lambda_2(c,p,\alpha)$, where

$$\lambda_1(c,p,\alpha) = \left(\frac{p-1}{p}\right)^{-\frac{1}{p}} \frac{\tilde{w}^{\frac{p-1}{\alpha+1}}}{\alpha+1} \tilde{w}^{\frac{-(p+\alpha)}{p}} \left[\frac{-\tilde{w}^2}{3p+\alpha-2}\left(1 - t^{\frac{3(p-1)}{\alpha+1}+1}\right) + \tilde{w}\left(\frac{1+c}{2p+\alpha-1}\right)\left(1 - t^{\frac{2(p-1)}{\alpha+1}+1}\right) - \left(\frac{c}{p+\alpha}\right)\left(1 - t^{\frac{p-1}{\alpha+1}+1}\right)\right]^{-\frac{1}{p}} dt,$$

and

$$\lambda_2 = \left(\frac{p}{p-1}\right)^{-\frac{1}{p}} \int_0^1 [H(\rho_1) - H(\rho_1 t)]^{-\frac{1}{p}} dt.$$

So, by Theorem 2, we have

i. If $\lambda > \lambda_1$, then the problem (1) admits a unique positive solution in A^+,

ii. If $\lambda = \lambda_1$, then the problem (1) admits two solutions, one in A^+ and the other one in B^+,

iii. If $\lambda < \lambda_2$, then the problem (1) admits no solution,

iv. If $\lambda_2 < \lambda < \lambda_1$, then the problem (1) admits exactly two solutions in A^+,

v. If $\lambda = \lambda_2$, then the problem(1) admits a unique solution in B^+.

Proof of assertion (B)

Assume that $p > 2$, by Lemma 2 and Lemma 4, we have

- $\lim_{\rho \to \frac{\tilde{w}^{\frac{p-1}{\alpha+1}}}{\alpha+1}} \tilde{T}(p,\lambda,c,\rho) = M(c,p,\alpha)$,
- $\lim_{\rho \to \frac{1}{\alpha+1}} \tilde{T}(p,\lambda,c,\rho) = M_1\left(\frac{1}{\alpha+1}, p\right)$.
- The function $\rho \mapsto \tilde{T}(p,\lambda,c,\rho)$ is strictly decreasing on (ρ_*, ρ_1) and strictly increasing on $\left(\rho_1, \frac{1}{\alpha+1}\right)$.
- There exists $\tilde{\alpha}$ such that for all $\alpha > \tilde{\alpha}$ we have, $M(c,p,\alpha) < \frac{1}{2}$ and $M_1(c,p,\alpha) > \frac{1}{2}$.

On the other hand, we have

- $M(c,p,\alpha) > \frac{1}{2}$ if $\lambda > \lambda_*$, where

$$\lambda_* = 2^p \left(\frac{p}{p-1}\right)\left(\frac{\alpha+1}{\tilde{w}^{\frac{p-1}{\alpha+1}}}\right)^{-p} \tilde{w}^{-(p+\alpha+1)} \frac{(2p+\alpha-1)^2}{(1+c)(\alpha+1)} B\left(\frac{\alpha+1}{2p+\alpha-1}, \frac{p-1}{p}\right)^{-p}.$$

- $M_1(c,p,\alpha) > \frac{1}{2}$ if $\lambda > \lambda_{**}$, where

$$\lambda_{**} = 2^p \left(\frac{p}{p-1}\right)(\alpha+1)^{\frac{1}{p}} \frac{(2p+\alpha-1)^2}{(1+c)(\alpha+1)} B\left(\frac{1}{2p+\alpha-1}, \frac{p-1}{p}\right)^{-p}$$

.

Now, if we set by definition

$$\lambda_3(c,p,\alpha) = \int_0^{\frac{1}{\alpha+1}} \left(\frac{p}{p-1}\right)^{-\frac{1}{p}} \left(\frac{1}{\alpha+1} - u\right)^{-\frac{2}{p}} \left[\theta\left(\frac{1}{\alpha+1}, u\right)\right]^{-\frac{1}{p}} du$$

then by Theorem 2, we have

i. If $\lambda < \lambda_2$, then the problem (1) admits no solution,

ii. If $\lambda = \lambda_2$,then the problem (1) admits a unique solution in A^+,

iii. If $\lambda > \lambda_3$ and $\alpha > \tilde{\alpha}$, then the problem (1) admits a unique solution in A^+,

iv. if $\lambda > \max(\lambda_*, \lambda_{**})$,then the problem (1) admits exactly two solutions in A^+.

6. Physical Interpretation and Numerical Illustration

a. Physical interpretation of the parameters

The mathematical model (1) involves four dimensionless parameters: $p > 1$, $\alpha \in \mathbb{R}$, $c > 0$ and $\lambda > 0$. Each of them carries a specific physical meaning,may describe phenomena such as:

The exponent p (nonlinear diffusion). The operator $\varphi_p(y) = |y|^{p-2}y$ defines the p-Laplacian. For $p = 2$ one recovers the standard linear Laplacian. When $p > 2$, the diffusion is *shear- thinning*: the flux becomes very small for small concentration gradients. This behaviour is typical of non- Newtonian fluids (e.g. polymers, blood) or of biological populations whose motility decreases at low densities. For $1 < p < 2$ the medium is *shear- thickening* (diffusion accelerates with the gradient).

The exponent α (concentration- dependent mobility). The factor $\varphi_p(u^\alpha)$ makes the diffusion coefficient depend on the concentration u itself. If $\alpha > 0$, the mobility increases with u: the more particles (or bacteria) are present, the faster they move. This is observed in some porous media flows and in bacterial quorum sensing, where motility is only triggered above a critical population. If $\alpha < 0$, mobility decreases with concentration (crowding or adhesion effects). A large $|\alpha|$ amplifies the nonlinearity, leading to an almost “all- or- nothing” behaviour.

The threshold c (bistable reaction). The source term $\lambda(u^{p-1} - u^{2p-2})(u^{p-1} - c)$ changes sign at $u = c^{1/(p-1)}$. For simplicity we refer to c as the *critical concentration*. Below c the reaction term is negative (decay or consumption); above c it becomes positive (growth or aggregation). In applications, c models a quorum sensing threshold in bacteriology, a gelation threshold in polymer physics, or a phase- transition value in material science.

The intensity λ (reaction strength). λ multiplies the whole reaction term and measures the *strength* of the nonlinear interaction. For small λ diffusion dominates and the concentration tends to homogenise. For large λ the reaction prevails, favouring phase separation and the coexistence of two distinct concentration levels – a phenomenon known as *bistability*.

Combined effect. The interplay between the degenerate diffusion (p, α) and the bistable source (c, λ) determines the exact number of positive stationary solutions. As shown in the main theorem, for $p > 2$ and sufficiently large λ (more precisely $\lambda > \max(\lambda_*, \lambda_{**})$) the problem admits exactly two positive solutions, corresponding to a stable low- concentration state and an unstable high- concentration state.

b. Numerical illustration

In this section, we provide a concrete physical example illustrating the exact multiplicity of positive solutions for a class of p-Laplacian quasilinear boundary value problems with Dirichlet conditions, in the case $p > 2$. Using the theoretical framework of the quadrature method (time-mapping), we select parameters $\alpha = 10$, $c = 0.2$, $p = 3$ and $\lambda = 1500$, which satisfy $\lambda > \max(\lambda_*, \lambda_{**})$. According to the main theorem, the problem admits exactly two positive solutions in A^+: one stable low-concentration solution and one unstable high-concentration solution. This models bistability phenomena in porous media flow or bacterial quorum sensing.

We choose the following parameters, satisfying the conditions of Theorem 1, part (B), case (iv).

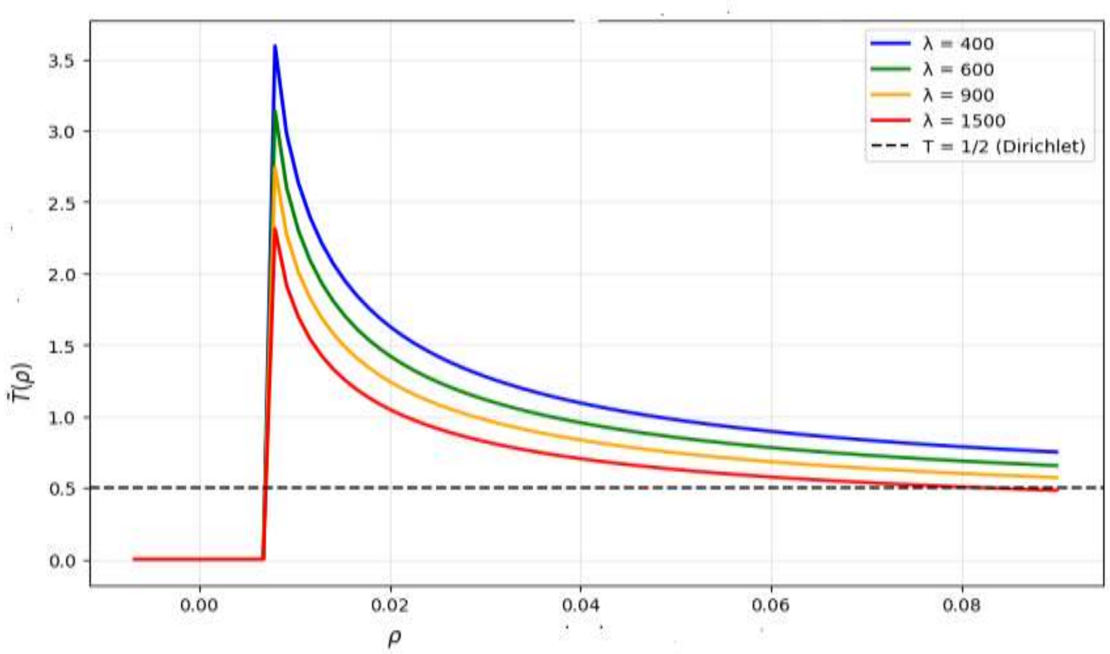
- $p = 3$ (cubic Laplacian, strong nonlinearity),
- $\alpha = 10$ (very high sensitivity to density),
- $c = 0.2$ (low threshold, below which diffusion is active),

- $\lambda = 1500$ (large reaction intensity).

According to the article, for $p > 2$ there exist λ_*, λ_{**} such that if $\lambda > \max(\lambda_*, \lambda_{**})$ then problem (1) admits exactly two positive solutions in A^+ (solutions with $u'(0) > 0$). Numerical estimation of the time- map (see Figure 1) shows that $\max(\lambda_*, \lambda_{**})$ lies between 900 and 1500; hence $\lambda = 1500$ satisfies the required inequality.

7. Discussion

The figure provides a clear numerical validation of the exact multiplicity results established in Theorem 1 for the case $p > 2$. It illustrates the transition from the absence of solutions (small λ) to the existence of exactly two solutions (sufficiently large λ) through the variation of the time-map.These results are consistent with the physical interpretation: when the reaction intensity λ is small, diffusion dominates and no non-trivial positive solution exists. For intermediate values of λ, two solutions appear (one stable, one unstable). For very large values of λ, the reaction prevails and a single solution survives.



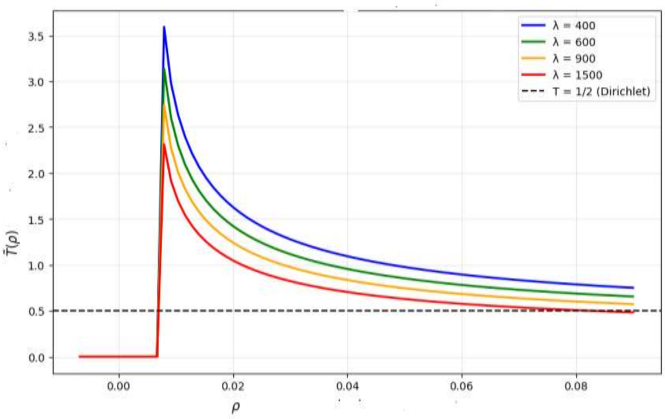


Figure 1: *Graph of the time-map $T(\rho)$ and its intersection with the level $\frac{1}{2}$*

8. Conclusions

In this work, we have established the exact number of positive solutions for a quasilinear boundary value problem with a density-dependent diffusion operator and a bistable nonlinearity. We have highlighted a transition in the solution structure when the parameter p crosses the value 2. The obtained results generalize those in the existing literature and open perspectives for the study of analogous problems in higher dimensions or with other boundary conditions.

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