



Multiplicity of Positive Steady States for a Semipositone p -Laplacian Neumann Problem Arising in Radiolysis

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Abstract:

In this paper, we study the existence and multiplicity of positive solutions for the one-dimensional p -Laplacian Neumann boundary value problem. The nonlinearity is semipositone since $f(0) < 0$. Using the quadrature method (time-map method), we transform the problem into an algebraic equation involving a time-map function and establish conditions for the existence, uniqueness, and multiplicity of positive solutions depending on the parameter λ . In particular, we prove the existence of at least three positive solutions, and more generally, at least $2n + 1$ positive solutions for suitable ranges of λ . Numerical examples and direct simulations for both cases $1 < p < 2$ and $p > 2$ are presented to illustrate the theoretical results. A physical interpretation is also provided within the framework of radiation chemistry: the problem describes the steady-state spatial distribution of reactive species generated by the radiolysis of confined aqueous systems under continuous irradiation, where the semipositone reaction term reflects the balance between radiation-induced production, recombination, and saturation, and the parameter λ encodes the dose rate. The multiple positive solutions correspond to distinct equilibrium concentration profiles attainable under identical irradiation conditions, a feature relevant to the accumulation of radiolytic products such as H_2 and H_2O_2 in sealed irradiated systems. Our results extend existing works on semipositone Neumann problems and provide a clearer description of the bifurcation structure of positive solutions for the p -Laplacian operator.

1. Introduction

We consider the nonlinear p -Laplacian boundary value problem

$$\begin{cases} -(\varphi_p(u'))' = \lambda f(u), & x \in (0,1), \\ u'(0) = u'(1) = 0, \end{cases} \quad (1.1)$$

where $\lambda > 0$ is a real parameter, $p > 1$, and $\varphi_p(y) = |y|^{p-2}y$ for $y \in \mathbb{R}$. The nonlinearity is given by

$$f(u) = a\varphi_p(u) - b(\varphi_p(u))^2 - c,$$

where $a > b > c > 0$. Moreover, since $f(0) < 0$, the problem is semipositone. Neumann boundary value problems of this type arise in several areas of mathematical physics and physical chemistry, in particular in nonlinear reaction–diffusion models. A physically relevant instance, which motivates the

present study and is developed in Section 4.3, is the radiolysis of water and aqueous solutions under ionizing radiation. When such a medium is irradiated by γ -rays, X-rays, or accelerated electrons, a variety of reactive species the homogeneous Neumann conditions model impermeable (sealed) boundaries, the parameter λ encodes the dose rate, and the exponent p accounts for gradient-dependent diffusion in non-standard irradiated media such as gels, polymer matrices, or porous solid–liquid systems. For the classical case $p = 2$, problems of this kind have been extensively studied using the quadrature (or time-map) method; see [1,2]. This approach, originally introduced by Opial [3] and later developed by Laetsch [4], Smoller and Wasserman [5,6], has proven to be an effective tool for studying the existence and multiplicity of solutions. For the p -Laplacian case, the method was extended by Manasevich and Zanolin [7], and has since been widely used in nonlinear analysis; see

also [8–10]. The main idea of the quadrature method is to reduce problem (1.1) to an algebraic equation of the form

$$G(s) = \lambda^{1/p},$$

where G is the associated time-map function. In this paper, we investigate the existence and multiplicity of positive solutions for problem (1.1). In particular, we establish intervals of the parameter λ for which at least three positive solutions exist, and more generally, $2n + 1$ positive solutions. We complement the theoretical analysis with explicit numerical examples and direct simulations for both cases $1 < p < 2$ and $p > 2$. These simulations confirm the validity and sharpness of the theoretical results. Recently, nonlinear p -Laplacian boundary value problems with Neumann conditions have attracted considerable attention. Existence, multiplicity, and uniqueness results have been established using variational methods, fixed-point techniques, and monotone operator theory; see [11,12]. More recent developments on p -Laplacian and semipositone boundary value problems, including fractional and Hadamard-type operators, can be found in [15–17]. Finally, a physical interpretation of the model within radiation chemistry is provided in Section 4.3: the p -Laplacian operator describes nonlinear diffusion of reactive species, while the semipositone reaction term reflects the coexistence of consumption, radiation-induced production, and saturation phenomena characteristic of radiolytic kinetics. The paper is organized as follows. Section 2 presents some preliminary results, Section 3 contains the main existence and multiplicity theorems, and Section 4 illustrates the theory with numerical examples, direct simulations, and the physical interpretation.

2. Preliminary Lemmas

Lemma 2.1. If $u(x)$ is a solution of (1.1), then $u(1 - x)$ is also a solution of (1.1).

Lemma 2.2. Any constant $u \in \mathbb{R}$ such that $f(u) = 0$ is a solution of (1.1).

Lemma 2.3. Let $u(x)$ be a solution of (1.1). If there exists $x_0 \in [0, 1]$ such that $u'(x_0) = 0$, then u is symmetric with respect to x_0 , that is,

$$u(x_0 - z) = u(x_0 + z), \quad \forall z \in [0, \min\{x_0, 1 - x_0\}].$$

Proof. Define $w_1(z) = u(x_0 - z)$ and $w_2(z) = u(x_0 + z)$. Then both w_1 and w_2 satisfy the same initial value problem. By uniqueness, we obtain $w_1(z) = w_2(z)$ for all admissible z .

3. Main Results

Theorem 3.1. Let $f(u) = a\varphi_p(u) - b(\varphi_p(u))^2 - c$, where $a > b > c$ are real constants. Assume that $f(0) < 0$, $f'(u) > 0$ for $u > 0$, and $\lim_{u \rightarrow +\infty} f(u) > 0$. Let

$$\beta = \left(\frac{a + \sqrt{a^2 - 4bc}}{2b} \right)^{\frac{1}{p-1}}$$

be the unique positive zero of f , and let θ be the unique zero of $F(s) = \int_0^s f(t) dt$ with $\theta > \beta$. Assume that the interval

$$S = \left(\frac{\theta^p}{-pF(\beta)}, \pi^2 \lambda_2 \right)$$

is nonempty. Then, for all $\lambda \in S$, problem (1.1) admits at least three positive solutions.

Theorem 3.2. Let f satisfy the same assumptions as in Theorem 3.1. Let $n \in \mathbb{N}$, and assume that the interval

$$S_n = \left(\frac{\theta^p}{-pF(\beta)}, \pi^{2n} p \lambda_2 \right)$$

is nonempty. Then, for all $\lambda \in S_n$, problem (1.1) admits at least $2n + 1$ positive solutions.

Theorem 3.3. Assume the hypotheses of Theorem 3.2. Let $\lambda > 0$. If there exists $\alpha \in (0, \beta)$ such that $T_+(\alpha) = \lambda^{1/p}$, then problem (1.1) admits a unique positive solution $u(x)$ satisfying $u(0) = \alpha$, $u(1) = \gamma$, where $\gamma = \gamma(\alpha)$ is defined by $F(\gamma) = F(\alpha)$, and $u'(x) > 0$ on $(0, 1)$. Moreover, there exist constants $0 < \lambda_2 < \lambda_1 < \infty$ such that:

1. If $\lambda > \lambda_1$, then problem (1.1) has no solution.
2. If $\lambda_2 < \lambda < \lambda_1$, then problem (1.1) has a unique solution.
3. If $\lambda < \lambda_2$, then problem (1.1) has no solution.

where

$$\lambda_1 = \frac{\theta^p}{-pF(\beta)},$$

$$\lambda_2 = \frac{p^p(p-1)^{\frac{1-p}{p}}}{f'(\beta)} \left[(\beta - \alpha)^{\frac{p-2}{p}} + (\gamma - \beta)^{\frac{p-2}{p}} \right]^p.$$

3.1. Proof of Theorems 3.1 and 3.3

We apply the quadrature technique to problem (1.1). First, multiplying equation (1.1) by $u'(x)$, we obtain

$$u'(x) \left(\varphi_p(u') \right)' = \lambda u'(x) f(u).$$

Hence,

$$u'(x) \left(\varphi_p(u') \right)' + \lambda u'(x) f(u) = 0. \quad (3.1)$$

This can be rewritten as

$$\frac{d}{dx} \left[\frac{|\varphi_p(u')|^p}{p} + \lambda F(u) \right] = 0, \quad x \in [0,1], \quad (3.2)$$

where $F(u) = \int_0^u f(s) ds$. Integrating (3.2), we obtain

$$\frac{|\varphi_p(u')|^p}{p} + \lambda F(u) = C_1, \quad (3.3)$$

where C_1 is a constant. Using the boundary conditions and assuming $u(0) = \alpha$ and $u(1) = \gamma$, we get

$$\lambda F(\alpha) = C_1 = \lambda F(\gamma).$$

Let θ denote the positive zero of F (see Figure 1). Then, for each $\alpha \in [0, \beta]$, there exists a unique $\gamma(\alpha) \in (\beta, \theta]$ such that $F(\alpha) = F(\gamma)$.

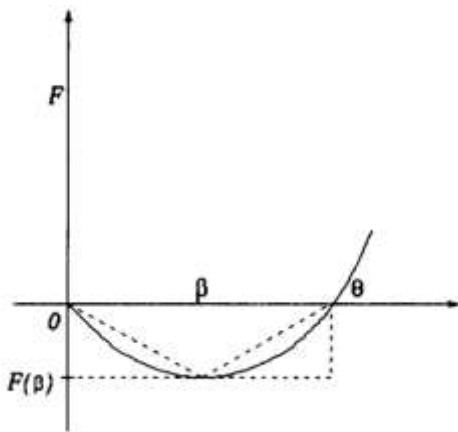


Figure 1: The behaviour of the function F

Thus, equation (3.3) becomes

$$\varphi_p(u') = (\lambda p)^{\frac{1}{p}} [F(\alpha) - F(u(x))]^{\frac{1}{p}}, \quad x \in [0,1]. \quad (3.4)$$

Integrating (3.4) over $[0, x]$, we obtain

$$\frac{1}{p^{1/p}} \int_{\alpha}^{u(x)} \frac{ds}{[F(\alpha) - F(s)]^{1/p}} = \lambda^{1/p} x. \quad (3.5)$$

Setting $x = 1$ in (3.5), we obtain

$$\lambda^{1/p} = \frac{1}{p^{1/p}} \int_{\alpha}^{\gamma(\alpha)} \frac{ds}{[F(\alpha) - F(s)]^{1/p}} = G(\alpha). \quad (3.6)$$

Thus, solving the boundary value problem reduces to solving the equation

$$G(\alpha) = \lambda^{1/p}.$$

Let us introduce the change of variable

$$u = \alpha + t(\gamma(\alpha) - \alpha), \quad du = (\gamma(\alpha) - \alpha) dt.$$

Then,

$$G(\alpha) = (\gamma(\alpha) - \alpha) \int_0^1 [F(\alpha) - F(\alpha + t(\gamma(\alpha) - \alpha))]^{-1/p} dt.$$

The derivative of $G(\alpha)$ is given by

$$\frac{dG}{d\alpha} = -\frac{1}{p} \int_{\alpha}^{\gamma(\alpha)} \frac{H_1(\alpha) - H_1(s)}{[F(\alpha) - F(s)]^{\frac{p+1}{p}}} ds + \frac{d\gamma}{d\alpha} \cdot \frac{1}{p} \int_{\alpha}^{\gamma(\alpha)} \frac{H_2(\gamma) - H_2(s)}{[F(\alpha) - F(s)]^{\frac{p+1}{p}}} ds, \quad (3.7)$$

where

$$H_1(s) = pF(s) - (\gamma(\alpha) - s)f(s),$$

$$H_2(s) = pF(s) + (\alpha - s)f(s).$$

A detailed analysis shows that $H_1'(s) < 0$ and $H_1''(s) > 0$ on $(\alpha, \gamma(\alpha))$, which implies $H_1(s) < 0$. Similarly, $H_2'(s) < 0$ and $H_2''(s) > 0$ on $(\alpha, \gamma(\alpha))$. Therefore,

$$H_1(\alpha) - H_1(s) \geq 0, \quad H_2(\gamma) - H_2(s) \geq 0,$$

and since $\frac{d\gamma}{d\alpha} < 0$, we deduce that

$$\frac{dG}{d\alpha} \leq 0, \quad \alpha \in [0, \beta].$$

Hence, G is decreasing.

To prove Theorem 3.1, we estimate $G(0)$ and $\lim_{\alpha \rightarrow \beta^-} G(\alpha)$. First, we show that

$$\frac{\theta^p}{-pF(\beta)} \leq [G(0)]^p \leq \left(\frac{p}{p-1} \right)^{p-1} \cdot \frac{\theta^p}{-F(\beta)}. \quad (3.8)$$

Next, we establish

$$\frac{p-1}{f'(\beta)} \left[(\beta - \alpha)^{\frac{p-2}{p}} + (\gamma - \beta)^{\frac{p-2}{p}} \right]^p \leq \lim_{\alpha \rightarrow \beta^-} [G(\alpha)]^p \leq \frac{p^p(p-1)^{1-p}}{f'(\beta)} \left[(\beta - \alpha)^{\frac{p-2}{p}} + (\gamma - \beta)^{\frac{p-2}{p}} \right]^p. \quad (3.9)$$

Proof of inequality (3.8).

$$\lim_{\alpha \rightarrow 0^+} G(\alpha) = \frac{1}{p^{1/p}} \int_0^\theta \frac{ds}{[-F(s)]^{1/p}} = G(0).$$

Since $F''(s) = f'(s) > 0$ for $s \in (0, (a/2b)^{\frac{1}{p-1}})$, we have

$$-F(s) \geq \begin{cases} -\frac{F(\beta)}{\beta} s, & 0 < s \leq \beta, \\ -\frac{F(\beta)}{\theta - \beta} (\theta - s), & \beta < s < \theta. \end{cases}$$

Hence,

$$\frac{1}{[-F(s)]^{1/p}} \leq \begin{cases} \left(\frac{\beta}{-F(\beta)s}\right)^{1/p}, & 0 < s \leq \beta, \\ \left(\frac{\theta - \beta}{-F(\beta)(\theta - s)}\right)^{1/p}, & \beta < s < \theta. \end{cases} \quad (3.10)$$

Rewriting $G(0)$ from (3.6) and bounding from above:

$$G(0) \leq \theta \left(\frac{p}{p-1}\right)^{\frac{p-1}{p}} \frac{1}{[-F(\beta)]^{1/p}}. \quad (3.11)$$

From the behavior of F (see Figure 1), $-F(s) \leq -F(\beta)$ for all $s \in [0, \theta]$, hence

$$G(0) \geq \frac{1}{p^{1/p}} \cdot \frac{\theta}{[-F(\beta)]^{1/p}}.$$

From (3.10) and (3.11), we obtain (3.8).

Proof of inequality (3.9). We can observe that

$$G(\alpha) \leq \frac{1}{p^{1/p}} \int_\alpha^{\gamma(\alpha)} \frac{ds}{[Z(s)]^{1/p}}, \quad (3.12)$$

where $Z(s)$ is defined as

$$Z(s) = \begin{cases} \frac{F(\alpha) - F(\beta)}{\beta - \alpha} (s - \alpha), & \alpha < s < \beta, \\ \frac{F(\gamma) - F(\beta)}{\gamma - \beta} (\gamma - s), & \beta < s < \gamma. \end{cases} \quad (3.13)$$

Furthermore,

$$\int_\alpha^{\gamma(\alpha)} \frac{ds}{[Z(s)]^{1/p}} = \frac{p}{p-1} \left[\frac{\beta - \alpha}{[F(\alpha) - F(\beta)]^{1/p}} + \frac{\gamma - \beta}{[F(\gamma) - F(\beta)]^{1/p}} \right]. \quad (3.14)$$

When $\alpha \rightarrow \beta^-$ and $\gamma \rightarrow \beta^+$, we have

$$\begin{aligned} \lim_{\alpha \rightarrow \beta^-} \left(\frac{p}{p-1}\right)^p \frac{(\beta - \alpha)^p}{F(\alpha) - F(\beta)} &= \frac{p^{p+1}(p-1)^{1-p}}{f'(\beta)} (\beta - \alpha)^{p-2} \\ &= \begin{cases} 0 & \text{if } p > 2, \\ \frac{8}{f'(\beta)} & \text{if } p = 2, \\ +\infty & \text{if } 1 < p < 2. \end{cases} \end{aligned} \quad (3.15)$$

$$\begin{aligned} \lim_{\gamma \rightarrow \beta^+} \left(\frac{p}{p-1}\right)^p \frac{(\gamma - \beta)^p}{F(\gamma) - F(\beta)} &= \frac{p^{p+1}(p-1)^{1-p}}{f'(\beta)} (\gamma - \beta)^{p-2} \\ &= \begin{cases} 0 & \text{if } p > 2, \\ \frac{8}{f'(\beta)} & \text{if } p = 2, \\ +\infty & \text{if } 1 < p < 2. \end{cases} \end{aligned}$$

From equations (3.12), (3.13), and (3.14):

$$\begin{aligned} \lim_{\alpha \rightarrow \beta^-} G(\alpha) &\leq \frac{p(p-1)^{\frac{1-p}{p}}}{[f'(\beta)]^{1/p}} \left[(\beta - \alpha)^{\frac{p-2}{p}} \right. \\ &\quad \left. + (\gamma - \beta)^{\frac{p-2}{p}} \right]. \end{aligned} \quad (3.16)$$

Also,

$$G(\alpha) \geq \frac{1}{p^{1/p}} \cdot \frac{\beta - \alpha}{[F(\alpha) - F(\beta)]^{1/p}} + \frac{1}{p^{1/p}} \cdot \frac{\gamma - \beta}{[F(\gamma) - F(\beta)]^{1/p}}.$$

When $\alpha \rightarrow \beta^-$, using (3.13) and (3.14):

$$\begin{aligned} \lim_{\alpha \rightarrow \beta^-} G(\alpha) &\geq \frac{(p-1)^{1/p}}{[f'(\beta)]^{1/p}} \left[(\beta - \alpha)^{\frac{p-2}{p}} \right. \\ &\quad \left. + (\gamma - \beta)^{\frac{p-2}{p}} \right]. \end{aligned}$$

From (3.15) and (3.16), inequality (3.9) follows.

From (1.1), (3.15), and the monotonicity of G , we deduce the existence and multiplicity of solutions depending on λ . Let

$$\lambda_1 = \frac{\theta^p}{-pF(\beta)},$$

$$\lambda_2 = \frac{p^p(p-1)^{\frac{1-p}{p}}}{f'(\beta)} \left[(\beta - \alpha)^{\frac{p-2}{p}} + (\gamma - \beta)^{\frac{p-2}{p}} \right]^p.$$

Then:

1. If $\lambda > \lambda_1$, then problem (1.1) has no solutions.
2. If $\lambda_2 < \lambda < \lambda_1$, then problem (1.1) has a unique solution.
3. If $\lambda < \lambda_2$, then problem (1.1) has no solutions.

Finally, since the possible bifurcation points satisfy $\lim_{\alpha \rightarrow \beta^-} [G(\alpha)]^p = \pi^{2n} p \lambda_2$, and under the condition

$$\pi^2 \lambda_2 > \frac{\theta^p}{-pF(\beta)},$$

the range of $[G(\alpha)]^p$ contains the interval

$$S = \left(\frac{\theta^p}{-pF(\beta)}, \pi^2 \lambda_2 \right).$$

Therefore, by Theorem 3.3 and Lemmas 2.1–2.2, the problem admits three positive solutions for $\lambda \in S$.

3.2. Proof of Theorem 3.2

Since f is autonomous, every solution of problem (1.1) is symmetric with respect to its critical points (see Lemma 2.3). Therefore, to obtain positive solutions $v(x)$ with $(n-1)$ interior critical points, it suffices to consider solutions of the form

$$u(x) = v\left(\frac{x}{n}\right), \quad x \in [0, 1].$$

A direct computation yields

$$u'(x) = \frac{1}{n} v'\left(\frac{x}{n}\right),$$

and hence

$$\varphi_p(u'(x)) = \left(\frac{1}{n}\right)^{p-1} \varphi_p\left(v'\left(\frac{x}{n}\right)\right).$$

Differentiating, we obtain

$$-(\varphi_p(u'))' = \left(\frac{1}{n}\right)^p \left[-(\varphi_p(v'))'\right]\left(\frac{x}{n}\right).$$

Since v satisfies (1.1), we have $-(\varphi_p(v'))' = \lambda f(v)$, and thus

$$-(\varphi_p(u'))' = \frac{\lambda}{n^p} f(u(x)).$$

Moreover, the boundary conditions become

$$u'(0) = \frac{1}{n} v'(0), \quad u'(1) = \frac{1}{n} v'\left(\frac{1}{n}\right).$$

Therefore, u satisfies a Neumann problem of the same type as (1.1), but with the parameter λ replaced by $\frac{\lambda}{n^p}$. From Theorem 3.3, such a solution exists if and only if

$$\frac{\lambda}{n^p} \in \text{Range}([G(\alpha)]^p).$$

Furthermore, by symmetry (Lemma 2.1), $u(1-x)$ is also a solution, providing a second solution with the same number of interior critical points.

Now, assume that

$$S_n = (\lambda_1, n^p \pi^2 \lambda_2)$$

is nonempty and let $\lambda \in S_n$. Then, for every integer $m = 1, 2, \dots, n$, we have $\frac{\lambda}{m^p} \in (\lambda_1, \lambda_2)$. Hence, by Theorem 3.1 and Lemma 2.3, there exist at least two distinct positive solutions with $m-1$ interior critical points for each m . Together with the constant solution $u \equiv \beta$, this yields at least $2n+1$ positive solutions of problem (1.1) for $\lambda \in S_n$. This completes the proof of Theorem 3.2. $\square$

4. Application

We illustrate the theoretical results with two numerical examples corresponding to the two transport regimes of the diffusing reactive species, $1 < p < 2$ and $p > 2$. In the radiolysis interpretation of Section 4.3, u denotes the steady-state concentration of a reactive species, λ the (scaled) dose rate, and the exponent p the nature of the effective gradient-dependent diffusion in the irradiated medium. In both cases, the constant λ_2 is the bound arising from the limit $\alpha \rightarrow \beta^-$ in (3.9), evaluated at an admissible value $\alpha \in (0, \beta)$.

4.1. Numerical Examples

Case 1: $p > 2$ (gradient-thickening regime). Take $p = 3$, $a = 4$, $b = 1$, $c = 1$, so that $f(u) = 4u^2 - u^4 - 1$. The positive zeros of f are

$$\beta = \sqrt{2 - \sqrt{3}} \approx 0.5176, \quad \beta_1 = \sqrt{2 + \sqrt{3}} \approx 1.9319.$$

The primitive is $F(u) = \frac{4}{3}u^3 - \frac{1}{5}u^5 - u$, with $F(\beta) \approx -0.3401$ and $f'(\beta) = 8\beta - 4\beta^3 \approx 3.5863$. The first positive zero of F after β is $\theta \approx 0.9280$. Then

$$\lambda_1 = \frac{\theta^p}{-pF(\beta)} = \frac{(0.9280)^3}{-3 \times (-0.3401)} \approx 0.7833.$$

Taking $\alpha = 0.49 \in (0, \beta)$, the value $\gamma = \gamma(\alpha) \approx 0.5449$ is determined by $F(\gamma) = F(\alpha)$, so that $\beta - \alpha \approx 0.0276$ and $\gamma - \beta \approx 0.0273$. With $p^p(p-1)^{1-p} = 3^3 \cdot 2^{-2} = 6.75$,

$$\begin{aligned}\lambda_2 &= \frac{p^p(p-1)^{1-p}}{pf'(\beta)} \left[(\beta - \alpha)^{\frac{p-2}{p}} + (\gamma - \beta)^{\frac{p-2}{p}} \right]^p \\ &= \frac{6.75}{3 \times 3.5863} \left[(0.0276)^{\frac{1}{3}} + (0.0273)^{\frac{1}{3}} \right]^3 \approx 0.1379.\end{aligned}$$

The interval of Theorem 3.1 is

$$\begin{aligned}S &= (\lambda_1, \pi^2 \lambda_2) = (0.7833, \pi^2 \times 0.1379) \\ &\approx (0.7833, 1.361) \neq \emptyset.\end{aligned}$$

Hence, for all $\lambda \in (0.7833, 1.361)$, the model admits at least three positive steady-state profiles of the reactive species.

Case 2: $1 < p < 2$ (gradient-thinning regime). Take $p = 1.5$, $a = 3$, $b = 0.5$, $c = 1$. For $u \geq 0$, $\varphi_p(u) = u^{p-1} = \sqrt{u}$, so that $f(u) = 3\sqrt{u} - 0.5u - 1$. The positive zero β solves $3\sqrt{\beta} - 0.5\beta - 1 = 0$, giving $\beta \approx 0.1255$. Since $f(0) = -1 < 0$, the problem is semipositone. The primitive is

$$F(u) = 2u^{3/2} - \frac{1}{4}u^2 - u,$$

with $F(\beta) \approx -0.0405$ and $f'(\beta) = \frac{3}{2}\beta^{-1/2} - \frac{1}{2} \approx 3.7343$. The first positive zero of F after β is $\theta \approx 0.2872$. Then

$$\lambda_1 = \frac{\theta^p}{-pF(\beta)} = \frac{(0.2872)^{1.5}}{-1.5 \times (-0.0405)} \approx 2.5322.$$

Taking $\alpha = 0.01 \in (0, \beta)$, the value $\gamma = \gamma(\alpha) \approx 0.2691$ is determined by $F(\gamma) = F(\alpha)$. With $p^p(p-1)^{1-p} = (1.5)^{1.5} \cdot (0.5)^{-0.5} \approx 2.5981$,

$$\begin{aligned}\lambda_2 &= \frac{p^p(p-1)^{1-p}}{pf'(\beta)} \left[(\beta - \alpha)^{\frac{p-2}{p}} + (\gamma - \beta)^{\frac{p-2}{p}} \right]^p \\ &\approx 3.6594.\end{aligned}$$

The interval of Theorem 3.1 is

$$\begin{aligned}S &= (\lambda_1, \pi^2 \lambda_2) = (2.5322, \pi^2 \times 3.6594) \\ &\approx (2.5322, 36.12) \neq \emptyset.\end{aligned}$$

Hence, for all $\lambda \in (2.5322, 36.12)$, problem (1.1) admits at least three positive steady-state profiles.

4.2. Numerical Simulation

To complement the qualitative analysis, we now report direct numerical simulations of the time-map G and of the corresponding positive steady states. All computations use the same two parameter sets introduced above, so that the simulations validate the analytic constants β , θ , λ_1 and λ_2 .

Numerical method. The time-map was evaluated directly from its quadrature representation (3.6),

$$G(\alpha) = \left(\frac{1}{p}\right)^{1/p} \int_{\alpha}^{\gamma(\alpha)} \frac{ds}{[F(\alpha) - F(s)]^{1/p}},$$

where, for each $\alpha \in [0, \beta)$, the companion point $\gamma(\alpha) \in (\beta, \theta]$ is obtained by solving $F(\gamma) = F(\alpha)$. The integrable end-point singularities at $s = \alpha$ and $s = \gamma$ were removed by the affine substitution $s = \alpha + t(\gamma(\alpha) - \alpha)$, $t \in [0, 1]$, and the resulting integral was computed by adaptive Gauss–Kronrod quadrature. The zeros β and θ and the companion map $\gamma(\alpha)$ were located by a bracketed Brent root-finder. Given a value of λ in the admissible range, the spatial profile $u(x)$ of each non-constant solution was reconstructed from the first integral (3.4) by inverting the relation

$$\left(\frac{1}{p}\right)^{1/p} \int_{\alpha}^{u(x)} \frac{ds}{[F(\alpha) - F(s)]^{1/p}} = \lambda^{1/p} x$$

on a monotone arc and then extending the solution to $[0, 1]$ by the reflection symmetry of Lemmas 2.1 and 2.3. All routines were implemented in Python (NumPy/SciPy); the reported figures are fully reproducible from the stated parameters.

The time-map and the solution count. Figure 1 shows the computed time-map $G(\alpha)$ on $[0, \beta)$ for the two regimes. In both cases the numerics confirm the monotonicity $dG/d\alpha \leq 0$ established analytically. The two regimes differ sharply in their behavior as $\alpha \rightarrow \beta^-$, exactly as predicted by the limits (3.13)–(3.14): for $p > 2$ (Case 1, $p = 3$) the time-map remains bounded and $G(\alpha) \rightarrow 0^+$ as $\alpha \rightarrow \beta^-$, with maximal value $G(0) \approx 1.218$ attained at $\alpha = 0$ (so that $[G(0)]^p \approx \lambda_1 = 0.7833$); for $1 < p < 2$ (Case 2, $p = 1.5$) the time-map diverges, $G(\alpha) \rightarrow +\infty$ as $\alpha \rightarrow \beta^-$, in agreement with the $+\infty$ branch of (3.14), with $G(0) \approx 3.781$. The number of positive solutions for a given λ equals the number of intersections of the horizontal line $y = \lambda^{1/p}$ with the graph of G , augmented by the constant solution $u \equiv \beta$. For the representative values $\lambda = 1.0 \in S$ (Case 1) and $\lambda = 10 \in S$ (Case 2), shown as dashed lines in Figure 1, the line meets the graph and, together with the mirror solution of Lemma 2.1 and the constant state, yields the three positive solutions guaranteed by Theorem 3.1.

Multiple steady-state profiles. Figure 2 displays the three positive steady states for Case 1 at $\lambda = 1.0 \in S = (0.7833, 1.361)$: the spatially uniform

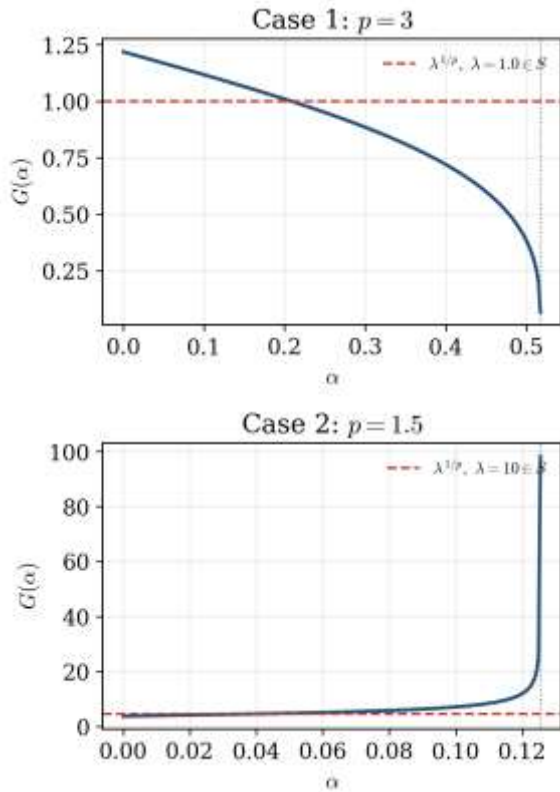


Figure 2. Computed time-map $G(\alpha)$ on $[0, \beta)$. Left: Case 1 ($p=3$), where G stays bounded and vanishes as $\alpha \rightarrow \beta^-$. Right: Case 2 ($p=1.5$), where G diverges as $\alpha \rightarrow \beta^-$. The dashed line is $\lambda^{1/p}$ for $\lambda \in S$; its intersection with the graph fixes the non-constant solution.

state $u \equiv \beta \approx 0.5176$, a monotone increasing profile $u(x)$ with $u(0) = \alpha = 0.40$, and its mirror image $u(1-x)$ provided by Lemma 2.1. All three satisfy the homogeneous Neumann conditions $u'(0) = u'(1) = 0$, and the non-constant profiles are symmetric about $x = 1/2$ in accordance with Lemma 2.3. The coexistence of these three profiles at a single value of λ is the multiplicity asserted by Theorem 3.1, here exhibited explicitly.

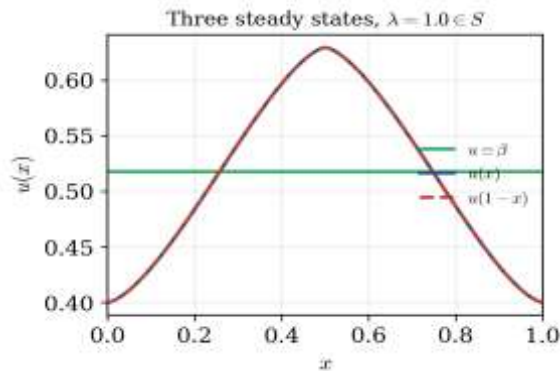


Figure 3. Three positive steady states for Case 1 at $\lambda = 1.0 \in S$: the constant state $u \equiv \beta$, a monotone profile

$u(x)$, and its mirror $u(1-x)$. In the radiolysis interpretation these are three distinct concentration profiles of the reactive species attainable under identical irradiation.

Bifurcation diagram. Figure 3 plots the boundary value $u(0) = \alpha$ of the non-constant branch against $\lambda = [G(\alpha)]^p$, together with the constant state $u \equiv \beta$ and the admissible interval S (shaded). The diagram makes the global solution structure transparent: in the band S the non-constant branch and the constant state coexist, and combined with the mirror branch this produces at least three positive solutions, while outside S the count drops in accordance with Theorem 3.3. The opposite monotonicity of the two branches—decreasing in λ for $p > 2$ and increasing for $1 < p < 2$ —reflects the contrasting end-point behavior of the time-map noted above and distinguishes the gradient-thickening ($p > 2$) from the gradient-thinning ($1 < p < 2$) transport regime.

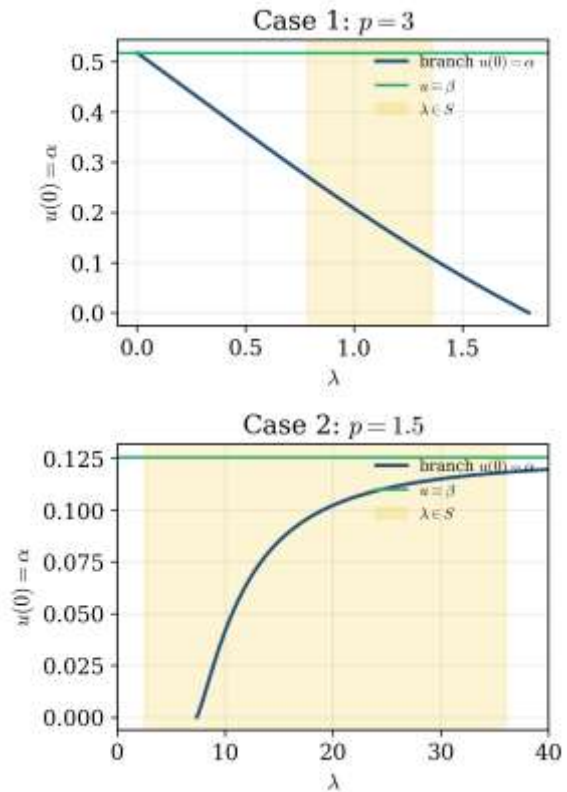


Figure 4. Bifurcation diagrams $u(0)=\alpha$ versus λ . Left: Case 1 ($p=3$). Right: Case 2 ($p=1.5$). The horizontal line is the constant state $u \equiv \beta$ and the shaded band is the interval S on which multiple positive solutions coexist.

The simulations confirm the three central predictions of the theory: the monotonicity of the time-map, the boundedness/divergence dichotomy at $\alpha \rightarrow \beta^-$ governed by the sign of $p - 2$, and the existence of an explicit parameter band S supporting at least three positive steady states.

4.3. Physical Interpretation: A Radiolysis Model

The p -Laplacian reaction–diffusion problem (1.1) admits a natural interpretation within the framework of radiation chemistry, specifically the radiolysis of water and aqueous solutions under continuous irradiation. This connection places the present mathematical study within the scope of radiation-induced chemical processes.

Physical setting. When water or an aqueous solution is exposed to ionizing radiation (γ -rays from ^{60}Co or ^{137}Cs sources, X-rays, or accelerated electrons), energy deposition along the radiation tracks generates a set of primary reactive species: hydrated electrons e_{aq}^- , hydroxyl radicals $\text{OH}^\bullet$, hydrogen atoms, and molecular products such as H_2 and H_2O_2 . In a closed, spatially confined system held under stable irradiation conditions, the competition between continuous radiolytic production and the various consumption pathways drives the concentration of a given reactive species toward a steady-state spatial profile. It is this stationary regime that problem (1.1) describes, with $u(x)$ representing the local concentration of the reactive species along a one-dimensional cell of normalized length, and λ encoding the irradiation intensity (dose rate) together with the geometric scaling of the domain.

Boundary conditions. The homogeneous Neumann conditions $u'(0) = u'(1) = 0$ correspond to a vessel with impermeable walls across which no net flux of the species occurs—a sealed irradiation cell with no mass exchange at the boundaries. This is the chemically relevant idealization for a confined irradiated sample, as opposed to the Dirichlet conditions appropriate for absorbing or reservoir-controlled boundaries.

The diffusion operator. In the classical case $p = 2$, the operator reduces to the ordinary Laplacian and (1.1) becomes the standard Fickian reaction–diffusion equation routinely used in radiolysis kinetics modeling. The general p -Laplacian $-(\varphi_p(u'))'$ extends this description to media in which the effective diffusivity of the reactive species depends nonlinearly on the local concentration gradient. Such gradient-dependent transport arises in non-standard media of practical interest in radiation chemistry—radiation-sensitive gels (as used in gel dosimetry), polymer matrices, and porous or heterogeneous solid–liquid systems—where the mobility of radiolytic products departs from the simple Fickian regime. The case $p = 2$ is recovered as the physical reference, while $p \neq 2$ models shear-thinning-type ($1 < p < 2$) or shear-thickening-type ($p > 2$) transport behavior of the diffusing species.

The semipositone reaction term. The nonlinearity

$$f(u) = a\varphi_p(u) - b(\varphi_p(u))^2 - c, \quad a > b > c > 0,$$

captures the three-stage balance characteristic of radiolytic kinetics. The constant term $-c$ represents a background consumption of the species (recombination, scavenging by impurities, or reaction with dissolved solutes) that proceeds even at vanishing concentration; this is precisely why $f(0) = -c < 0$, making the problem semipositone. Physically, this reflects the well-documented fact that in the absence of local production a reactive species is continuously removed through back-reactions and impurity-driven loss pathways, which dominate the early radiolytic balance and govern the resulting radical concentrations. The linear-in- φ_p term $a\varphi_p(u)$ represents radiation-driven production (activation, generation along the tracks), while the quadratic term $-b(\varphi_p(u))^2$ accounts for second-order self-recombination for instance $\text{OH}^\bullet + \text{OH}^\bullet \rightarrow \text{H}_2\text{O}_2$ which saturates and inhibits growth at high concentration. Thus f is negative for small u (net consumption), positive at intermediate u (net production), and negative again for large u (saturation and recombination-dominated loss), reproducing the production–recombination–saturation structure of radiolysis.

Equilibrium and multiplicity. The positive zero β of f corresponds to the homogeneous steady state at which radiolytic production and consumption balance exactly, yielding a spatially uniform concentration of the reactive species throughout the cell. The multiple positive solutions established in Theorems 3.1 and 3.3 reveal that, under fixed irradiation conditions (fixed λ), the system may also support spatially non-uniform steady states. These correspond to heterogeneous concentration profiles generated by the competition between nonlinear diffusion and the radiolytic reaction term—for example localized regions of elevated reactive-species concentration coexisting with depleted regions within the same irradiated sample.

This multiplicity has a concrete consequence in radiation chemistry: for one and the same dose rate, an irradiated confined system may settle into several distinct concentration regimes depending on initial conditions, a feature relevant to the predictability and control of radiolytic product accumulation (notably H_2 and H_2O_2) in sealed irradiated systems such as those encountered in nuclear coolant chemistry and waste-storage canisters. The p -dependence further allows the model to interpolate between transport regimes characteristic of different irradiated media, making (1.1) a flexible qualitative

framework for the steady-state spatial behavior of reactive species under irradiation.

5. Conclusions

In this paper, we studied a semipositone Neumann boundary value problem involving the p -Laplacian operator. Using the time-map (quadrature) method, we transformed the problem into an equivalent algebraic equation and derived conditions ensuring existence and multiplicity of positive solutions. We established intervals of the parameter λ for which the problem admits at least three positive solutions and, more generally, $2n + 1$ positive solutions. The results are obtained through a detailed analysis of the time-map function and its monotonicity properties. Numerical examples and direct simulations for $1 < p < 2$ and $p > 2$ confirm the theoretical findings and illustrate the behavior of solutions in both shear-thinning and shear-thickening cases. These results extend previous works on semipositone Neumann problems and highlight the role of the p -Laplacian in generating multiple steady states.

Author Statements

- **Ethical approval:** The conducted research is not related to either human or animal use.
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References

- [1] Meciano, A. R., & Shivaji, R. (1993). Multiple positive solutions for a class of semipositone Neumann two-point boundary value problems. *Journal of Mathematical Analysis and Applications*, 178(1), 102–115.
- [2] Anuradha, V., Maya, C., & Shivaji, R. (1999). Positive solutions for a class of nonlinear boundary value problems with Neumann–Robin boundary conditions. *Journal of Mathematical Analysis and Applications*, 236(1), 94–124.
- [3] Opial, Z. (1961). Sur l'existence des solutions périodiques de l'équation différentielle $x'' + f(x, x')x' + g(x) = p(t)$. *Annales Polonici Mathematici*, 11.
- [4] Laetsch, T. W. (1970). The number of solutions of a nonlinear two-point boundary value problem. *Indiana University Mathematics Journal*, 20, 1–13.
- [5] Smoller, J., & Wasserman, A. (1981). Global bifurcation of steady-state solutions. *Journal of Differential Equations*, 39(2), 269–290.
- [6] Smoller, J., & Wasserman, A. (1989). On the monotonicity of the time-map. *Journal of Differential Equations*, 77(2), 188–201.
- [7] Manasevich, R., & Zanolin, F. (1993). Time-mapping and multiplicity of solutions for the one-dimensional p -Laplacian. *Nonlinear Analysis*, 21(4), 269–291.
- [8] Moghaddam, M. K., & Afrouzi, G. A. (2012). Existence of positive solutions for a class of p -Laplacian BVP with Neumann–Robin conditions. *International Journal of Mathematical Analysis*, 6(11), 517–526.
- [9] Derhab, M., & Sebbagh, H. (2012). Exact number of positive solutions for a class of quasilinear boundary value problems with a singular nonlinearity. *Communications on Applied Nonlinear Analysis*, 19(3), 103–125.
- [10] Derhab, M., & Sebbagh, H. (2013). Exact number of positive solutions for quasilinear boundary value problems with p -convex nonlinearity. *Filomat*, 27(3), 499–513.
- [11] Chen, T., & Ma, R. (2016). Positive solutions for a class of semipositone Neumann problems. *Boundary Value Problems*, 2016, Article 19, 1–12.
- [12] Jiang, J., & Sun, X. (n.d.). Existence of positive solutions for a class of p -Laplacian fractional differential equations with nonlocal boundary conditions. *Boundary Value Problems*.
- [13] Draganic, I. G. (2005). Radiolysis of water: A look at its origin and occurrence in nature. *Radiation Physics and Chemistry*, 72(2–3), 181–186.
- [14] Le Caër, S. (2011). Water radiolysis: Influence of oxide surfaces on H₂ production under ionizing radiation. *Water*, 3(1), 235–253.
- [15] F. Y. Deren, T. S. Cerdik, R. P. Agarwal, Existence criteria of positive solutions for fractional p -Laplacian boundary value problems, *Filomat* 34 (11) (2020) 3789–3799.
- [16] J. Xu, J. Jiang, D. O'Regan, Positive solutions for a class of p -Laplacian Hadamard fractional-order three-point boundary value problems, *Mathematics* 8 (3) (2020) 308.
- [17] E. Lopera, C. López, R. E. Vidal, Existence of positive solutions for a parameter fractional p -Laplacian problem with semipositone nonlinearity, preprint (2022), arXiv:2211.02790.
- [18] B. G. Ershov, A. V. Gordeev, A model for radiolysis of water and aqueous solutions of H₂, H₂O₂ and O₂, *Radiat. Phys. Chem.* 77 (2008) 928–935.

- [19] K. Iwamatsu, S. Sundin, J. A. LaVerne, Hydrogen peroxide kinetics in water radiolysis, *Radiat. Phys. Chem.* 145 (2018) 207–212.
- [20] J. S. Southworth, S. M. Pimblott, S. P. K. Koehler, H₂ production from the radiolysis of aqueous suspensions of ZnO nanoparticles by 5.5 MeV He²⁺ ions, *Radiat. Phys. Chem.* 197 (2022) 110189.