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**Research Article**



# Qualitative Behavior of Third Order Delay Difference Equations

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**Abstract:**

We examine the qualitative behaviour of solutions to a particular class of third-order nonlinear delay difference equations in this work. There are specified sufficient requirements for the solution to be oscillatory. An example is provided to highlight the main findings.

## 1. Introduction

In this research, we aim to explore the oscillatory characteristics exhibited by solution in nonlinear third order delay difference equation of the form

$$\Delta(w(\wp)(\Delta^2 z(\wp))^\gamma) = l(\wp)z^\gamma \qquad (1)$$

We make the following assumptions:

(i) The sequences $\{l(\wp)\}$ and $\{w(\wp)\}$ are positive real sequences,

(ii) $\gamma$ is expressed as ratios of positive odd integers and $\gamma \geq 1$,

(iii) $\mu(\wp) \leq \wp$ and $\mu(\wp) \to \infty$ as $\wp \to \infty$.

Additionally, we assume that:

$$Q(\wp, \wp_0) = \sum_{s=\wp_0}^{\wp-1} w^{-\frac{1}{\gamma}}(s) \to \infty \text{ as } \wp \to \infty \qquad (2)$$

A nontrivial real valued sequence $\{z(\wp)\}$ that fulfills (1) is considered a solution in the context of (1). Solutions that vanish identically in some region of infinity are excludedfrom consideration. If a solution z of (1) is neither finally positive nor negative, it is referred to as nonoscillatory; if it is, it is considered oscillatory. If every solution to an equation is oscillatory, then the equation itself is said to be oscillatory.

An exciting field of study that has spanned several decades has been the investigation of oscillation criteria for different kinds of differential equations. Several articles and books, including [1-9], have been written about this topic and provide insightful information. This paper's main goal is to present new oscillation criteria for (1) by comparing it to either a first-order linear delay difference equation or a third-order linear difference equation. The oscillating nature of these linear equivalents has been widely addressed in the literature.

## 2. Main Results

We use the following lemma to establish our findings. Lemma 2.1. Let $\{l(\wp)\}$ be a sequence of positive real numbers, $p$ is a positive real number and $f: R \to R$ is a continuous nondecreasing sequence and $zf(z) > 0$ for $z \neq 0$. If the first order delay difference inequality

$$\Delta x(\wp) + l(\wp) f(x(\mu(\wp)))) \leq 0$$

has an eventually positive solution, so does delay equation

$$\Delta x(\wp) + l(\wp) f(x(\mu(\wp))) = 0$$

(II)If the first order advanced difference inequality

$$\Delta x(\wp) - l(\wp) f(x(\mu(\wp))) \geq 0$$

has an eventually positive solution, then so does the corresponding advanced difference equation.

$$\Delta x(\wp) - l(\wp) f(\mu(\wp))) = 0$$

An extension of the discrete analogue of known outcomes is provided by this lemma. Refer to Lemma 6.2.2 in both [2] and [4]. The evidence is available right away.

Theorem 2.2. Assume $\gamma \geq 1$ and satisfying conditions (i)-(iii) along with (2), assume that there exists a nondecreasing sequence $\eta(\wp)$ and $\sigma(\wp) \ni$

$$\mu(\wp) \leq \sigma(\wp) \text{ for } \wp \geq \wp_1 \geq \wp_0 \quad (3)$$

$$\eta(\wp) < \wp, \eta\big(\mu(\wp)\big) \leq \mu(\wp), \zeta(\wp): \eta\Big(\eta\big(\mu(\wp)\big)\Big) > \wp \text{ for } \wp \geq \wp_0 \quad (4)$$

If the advanced equation

$$\Delta R(\wp) - \frac{1}{\gamma} l(\wp) \left[ \sum_{s=\eta(\mu(\wp))}^{\mu(\wp)} Q\big(s, \eta(s)\big) \right]^{\gamma} R\big(\zeta(\wp)\big) = 0 \quad (5)$$

is oscillatory and the first order delay difference eqauation

$$\Delta V(\wp) + \frac{1}{\gamma}(\theta^{\gamma}) l(\wp) \big(w^{-1}(\wp)\big) \mu^{\gamma}(\wp) \big(\sigma(\wp) - \mu(\wp)\big) \Big(V\big(\sigma(\wp)\big)\Big) = 0 \quad (6)$$

is oscillatory for all large $\theta \in (0,1)$ and $\wp \geq \wp_1 \geq \wp_0$, then (1) is oscillatory.

Proof. Let $z(\wp)$ be a nonoscillatory solution of (1) say $z(\wp) > 0$ and $z(\mu(\wp)) > 0$ for $\wp \geq \wp_1$, for some $\wp_1 \geq \wp_0$. It follows from (1) that

$$\Delta(w(\wp)(\Delta^2 z(\wp))^{\gamma}) = l(\wp) z^{\gamma}(\mu(\wp)) \geq 0$$

We have two cases:

(i) $z(\wp) > 0; \Delta z(\wp) \geq 0$ and $\Delta^2 z(\wp) \geq 0$

(ii) $z(\wp) > 0; \Delta z(\wp) \geq 0$ and $\Delta^2 z(\wp) \leq 0$

From (1), we see that

$$\Delta(w(\wp)(\Delta^2 z(\wp))^{\gamma}) = \Delta\left(w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp)\right)^{\gamma}$$

$\Delta$ - Derivative yields

$$\begin{aligned}
\Delta(w(\wp)(\Delta^2 z(\wp))^{\gamma}) &= \Delta\left(w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp)\right) \\
&= \gamma\left(w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp)\right)^{\gamma-1} \Delta\left(w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp)\right) \\
&= l(\wp) z^{\gamma}(\mu(\wp))
\end{aligned}$$

and so, we get

$$\Delta\left(w^{\frac{1}{\gamma}}(\wp)(\Delta^2 z(\wp))\right) = \frac{1}{\gamma}\left(w^{\frac{1}{\gamma}}(\wp)(\Delta^2 z(\wp))^{1-\gamma} l(\wp) z^{\gamma}(\mu(\wp))\right. \quad (7)$$

We start by thinking about Case (i) and the inequality.

$$\Delta\left(w^{\frac{1}{\gamma}}(\wp)(\Delta^2 z(\wp))\right) = \frac{1}{\gamma}\left(w^{\frac{1}{\gamma}}(\wp)(\Delta^2 z(\wp))^{1-\gamma} l(\wp) z^{\gamma}(\mu(\wp))\right. \quad (8)$$

Consequently,

$$\begin{aligned}
\Delta z(\wp) \geq \Delta z(\wp) - \Delta z(\eta(\wp)) &= \sum_{(s=\eta(\wp))}^{\wp-1} w^{\frac{-1}{\gamma}}(s)\left(w^{\frac{1}{\gamma}}(s)\Delta z(s)\right) \\
&\geq Q(n, \eta(\wp)) w^{\frac{1}{\gamma}}(\eta(\wp)) \Delta^2 z(\eta(\wp)).
\end{aligned}$$

Again we get,

$$z(\wp) \geq \sum_{s=\eta(\wp)}^{\wp-1} Q(s,\eta(s)) w^{\frac{1}{\gamma}} \eta(\wp) \Delta^2 z(\eta(s))$$

$$\geq w^{\frac{1}{\gamma}}\left(\eta\left(\eta(\wp)\right)\right) \Delta^2 z\left(\eta\left(\eta(\wp)\right)\right) \sum_{s=\eta(\wp)}^{\wp-1} Q\left(s,\eta(s)\right). \quad (9)$$

Using (9) in (8), we get

$$\Delta\left(w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp)\right) \geq \frac{1}{\gamma}\left(w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp)\right)^{1-\gamma} l(\wp)\left[w^{\frac{1}{\gamma}}(\zeta(\wp))\Delta^2 z(\zeta(\wp))\right]\left[\sum_{s=\eta(\mu(\wp))}^{\mu(\wp)-1} Q(s,\eta(s))\right]^{\gamma}$$

$$\geq \frac{1}{\gamma}\left[w^{\frac{1}{\gamma}}(\zeta(\wp))\Delta^2 z(\zeta(\wp))\right]^{1-\gamma}\left[w^{\frac{1}{\gamma}}(\zeta(\wp))\Delta^2 z(\zeta(\wp))\right]^{\gamma} l(\wp)$$
$$\left[\sum_{s=\eta(\mu(\wp))}^{\mu(\wp)-1} Q(s,\eta(s))\right]^{\gamma}$$

$$\geq \frac{1}{\gamma} l(\wp)\left[\sum_{s=\eta(\mu(\wp))}^{\mu(\wp)-1} Q(s,\eta(s))\right]^{\gamma} w^{\frac{1}{\gamma}}(\zeta(\wp))\Delta^2 z(\zeta(\wp))$$

Since $w(\wp)$ is a nondecreasing sequence and $\gamma > 1$, we have

$$\Delta R(\wp) = \Delta\left(w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp)\right) \geq \frac{1}{\gamma} l(\wp)\left[\sum_{s=\eta(\mu(\wp))}^{\mu(\wp)-1} Q(s,\eta(s))\right]^{\gamma} R(\zeta(\wp))$$

The related difference (5) likewise has a positive solution, which is a contradiction, as follows from Lemma 2.1(II).

Subsequently, we examine Case (ii). It is evident that a constant $0 \in (0,1)$ exists such that,

$$z\left(\mu(\wp)\right) \geq \theta\mu(\wp)\Delta z\left(\mu(\wp)\right) \text{ for } \wp \geq \wp_1 \quad (10)$$

Using this inequality in (7), we have

$$\Delta\left(w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp)\right) = \frac{1}{\gamma}\left(w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp)\right)^{1-\gamma} l(\wp) z^{\gamma}\left(\mu(\wp)\right) \quad (11)$$

and using (10) in (11), we obtain

$$\Delta\left(w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp)\right) \geq \frac{1}{\gamma}\left(w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp)\right)^{1-\gamma} l(\wp)[\theta\mu(\wp) z\left(\mu(\wp)\right)]^{\gamma} \quad (12)$$

Set $Y(\wp) = \Delta x(\wp)$, we see that

$$\Delta\left(w^{\frac{1}{\gamma}}(\wp)\Delta Y(\wp)\right) \geq \frac{1}{\gamma}\left(w^{\frac{1}{\gamma}}(\wp) Y(\wp)\right)^{1-\gamma} l(\wp)[\theta\mu(\wp) y(\mu(\wp))]^{\gamma}$$

or

$$\Delta\left(w^{\frac{1}{\gamma}}(\wp)\Delta Y(\wp)\right) \geq \frac{1}{\gamma}\theta^{\gamma} l(\wp)\left(w^{\frac{1}{\gamma}}(\wp)\right)^{1-\gamma} \mu^{\gamma}(\wp) Y\left(\mu(\wp)\right) \quad (13)$$

Now, for $m \geq p \geq \wp_1$ and see that

$$Y(p) - Y(m) \geq (m-p)(-\Delta Y(m))$$

Setting $p = \mu(\wp)$ and $m = \sigma(\wp)$, we have

$$Y\left(\mu(\wp)\right) \geq \left(\sigma(\wp) - \mu(\wp)\right)\left(-\Delta Y\left(\sigma(\wp)\right)\right) \quad (14)$$

Using this inequalites in (12), we have

$$\Delta\left(w^{\frac{1}{\gamma}}(\wp)\Delta Y(\wp)\right) \geq \frac{1}{\gamma}\theta^{\gamma} l(\wp)\left(w^{\frac{1}{\gamma}}(\wp)\right)^{1-\gamma} \mu^{\gamma}(\wp)(\sigma(\wp) - \mu(\wp))(-\Delta Y(\sigma(\wp)))$$

$$-\Delta V(\wp) \geq \frac{1}{\gamma}\theta^{\gamma} l(\wp) w^{-1}(\wp)\mu^{\gamma}(\wp)(\sigma(\wp) - \mu(\wp))(V(\sigma(\wp)))$$

where $V(\wp) = -w^{\frac{1}{\gamma}}(\wp)\Delta Y(\wp)$. Lemma 2.1(I) implies that there is a contradiction in that the comparable difference (6) likewise has a positive solution. The evidence is now complete.

**Corollary 2.3.** Assume $\gamma \geq 1$ and satisfying conditions (i)-(iii) along with (2), assume that $\exists a$ nondecreasing sequence $\eta(\wp)$ and $\sigma(\wp)$ such that (3) and (4) hold. If

$$\liminf_{\wp\to\infty} \sum_{s=\eta(\mu(\wp))}^{\mu(\wp)-1} l(s)w^{-1}(s)\mu^{\gamma}(s)\big(\sigma(s)-\mu(s)\big) = \infty \tag{15}$$

then it follows that (1) is oscillatory.

The comparison findings using third order linear difference inequalities are as follows.

**Theorem 2.4.** Assuming that there is a nondecreasing sequence $\eta(\wp)$ and $\sigma(\wp)$ such that (3) and (4) hold, let conditions (i)-(iii) and (2) hold. If the inequality

$$\Delta\left(w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp)\right) - \frac{1}{\gamma}\left[\sum_{s=\eta(\mu(\wp))}^{\mu(\wp)-1} Q\big(s,\eta(s)\big)\right]^{\gamma-1} l(\wp)z\big(\mu(\wp)\big) \geq 0 \tag{16}$$

has no eventually positive nondecreasing solution for $\wp \geq \wp_1 \geq \wp_0$ and the inequality

$$\Delta\left(w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp)\right) - \frac{1}{\gamma}\Big(\theta\mu(\wp)\big(\sigma(\wp)-\mu(\wp)\big)\Big(w^{\bar{\gamma}}\big(\sigma(\wp)\big)\Big)\Big)^{\gamma-1} l(\wp)z\big(\mu(\wp)\big) \geq 0 \tag{17}$$

has no eventually positive nonincreasing solution, then (1) is oscillatory.

Proof. Let $z(\wp)$ be a nonoscillatory solution of (1) say $z(\wp) > 0$ and $z(\mu(\wp)) > 0$ for some $\wp_1 \geq \wp_0$. Following the steps outlined in the Theorem 2.1 proof, we arrive at the situations (I) and (II). We examine case (I). It is clear from (9) that

$$w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp) \leq w^{\frac{1}{\gamma}}(\zeta(\wp))\Delta^2 z(\zeta(\wp)) \leq \left[\sum_{s=\eta(\mu(\wp))}^{\mu(\wp)-1} Q(s,\eta(s))\right]^{-1} z(\mu(\wp))$$

and, so

$$\left(w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp)\right) \geq \left[\sum_{s=\eta(\mu\mu(\wp))}^{\mu(\wp)-1} Q(s,\eta(\wp))\right]^{-1} (z(\mu(\wp)))^{1-\gamma}$$

Using this inequality in (9), we have

$$\Delta\left(w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp)\right) \geq \frac{1}{\gamma}\left[\sum_{s=\eta(\mu(\wp))}^{\mu(\wp)-1} Q(s,\eta(\wp))\right]^{\gamma-1} [z(\mu(\wp))]^{1-\gamma} l(\wp)z^{\gamma}(\mu(\wp)) \geq 0$$

or

$$\Delta\left(w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp)\right) - \frac{1}{\gamma}\left[\sum_{s=\eta(\mu(\wp))}^{\mu(\wp)-1} Q(s,\eta(s))\right]^{\gamma-1} [z(\mu(\wp))]^{1-\gamma} l(\wp)z(\mu(\wp)) \geq 0$$

By condition (16), we arrive desired contraction.

We then examine Case (II). Similar to the theorem 2.1 Case (II) proof, we get (10) and (14), with the last one having the following form:

$$\Delta z(\mu(\wp)) \geq (\sigma(\wp)-\mu(\wp))(-\Delta^2 z(\sigma(\wp)))$$

Thus, we have

$$z(\mu(\wp)) \geq \theta\mu(\wp)(\sigma(\wp)-\mu(\wp))w^{\frac{1}{\gamma}}(\sigma(\wp))\left(-w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\sigma(\wp))\right)$$

or

$$\left[\left[\theta\mu(\wp)(\sigma(\wp)-\mu(\wp))w^{\frac{-1}{\gamma}}(\sigma(\wp))\right]^{-1} z(\mu(\wp))\right] \geq \left(-a^{\frac{1}{\gamma}}(\sigma(\wp)\Delta^2 z(\sigma(\wp)))\right)$$

$$\left[\theta\mu(\wp)(\sigma(\wp)-\mu(\wp))w^{\frac{-1}{\gamma}}(\sigma(\wp))^{-1}z(\mu(\wp))\right]^{1-\gamma} \leq \left(-w^{\frac{1}{\gamma}}(\sigma(\wp)\Delta^2 z(\sigma(\wp))\right)^{1-\gamma}$$

$$\leq \left(-w^{\frac{-1}{\gamma}}(\wp)\Delta^2 z(\wp)\right)^{1-\gamma}.$$

Using this inequality in (11) we have

$$\Delta\left(w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp)\right) \geq \left[\left(\theta\mu(\wp)(\sigma(\wp)-\mu(\wp))w^{\frac{-1}{\gamma}}(\sigma(\wp))\right)^{-1} z(\mu(\wp))\right]^{1-\gamma} l(\wp)(z(\mu(\wp)))^{\gamma}$$

(or)

$$\Delta\left(w^{\frac{1}{\gamma}}(\wp)\Delta^2 z(\wp)\right) \geq \left[\left(\theta\mu(\wp)(\sigma(\wp)-\mu(\wp))w^{\frac{-1}{\gamma}}(\sigma(\wp))\right)^{-1} z(\mu(\wp))\right]^{1-\gamma} l(\wp)z(\mu(\wp))$$

We reached the intended contradiction via (16). The evidence is now complete.

**Example 2.5**. Consider the third order delay difference equation

$$\Delta\left(\frac{1}{\wp^3}(\Delta^2 z(\wp))^3\right) = \frac{1}{\wp^5} z^3\left(\frac{\wp}{4}\right) \qquad (18)$$

Here $\gamma = 3, w(\wp) = \frac{1}{\wp^3}, \mu(\wp) = \frac{\wp}{4}, l(\wp) = \frac{1}{\wp^5}$. We let $\sigma(\wp) = \frac{\wp}{2}$.

For condition (17) we find

$$\sum_{s=\frac{\wp}{2}}^{\wp-1} \frac{1}{s^5}\frac{s^6}{4^3}\left(\frac{s}{4}\right) \to \infty$$

as $\wp \to \infty$. At this point, Corollary 2.3's requirements are completely met. Eq.(18) is oscillatory in that case.

## Author Statements:

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